

VALUE OR GROWTH? PRICING OF IDIOSYNCRATIC CASH FLOW RISK  
WITH HETEROGENEOUS BELIEFS

A Dissertation

by

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Submitted to the Office of Graduate and Professional Studies of  
Texas A&M University  
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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May 2014

Major Subject: Finance

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## ABSTRACT

We study an equilibrium continuous-time exchange economy where idiosyncratic cash flow risks are priced via investors' heterogeneous beliefs. Investors perceive idiosyncratic cash flow risks differently through heterogeneous subjective mean growth rates on a firm's cash flows. This impacts equilibrium quantities. Our model shows that idiosyncratic cash flow shocks priced through belief differences can explain cross-sectional variation in stock returns and cash flows. Quantitative results show that a value premium arises, as value stocks have higher idiosyncratic cash-flow volatilities, lower average cash flows, and higher belief differences, which is empirically supported. A growth premium prevails without belief differences.

## DEDICATION

To my parents, Soonki Jhang and Donghee Jo

To my parents-in-law, Chang Geun Park and Wonjah Lee

&

To my wife, Heesun Park

&

Finally, to my daughter, Yuna Jhang

## ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my advisors, Prof. Hwagyun Kim and Prof. Michael Gallmeyer (University of Virginia, outside member) for their continuous support of my Ph.D study and research, for their patience, motivation, enthusiasm, and immense knowledge. Their guidances helped me in all the time of research and writing of this thesis. I could not have imagined having better advisors and mentors for my Ph.D study.

Beside my advisors, I would like to thank the rest of my thesis committee: Prof. Shane Johnson, Prof. Yong Chen, Prof. James Kolari and Prof. Scott Lee (University of Las Vegas at Nevada) for their encouragement, insightful comments, and hard questions.

My sincere thanks also goes to Emmanuel Alanis, Imran Haque and other Ph.D. students in the Finance Department at Texas A&M University, for helping me in many ways.

I thank my fellow Dr. Kibeom Binh (Myoungji University), Dr. Hyunsoo Choi (Singapore Management University) and many others for the stimulating discussions. Also I am grateful to Prof. Christopher Jones, Prof. Fernando Zapatero (University of Southern California), and Prof. Donghyun Ahn (Seoul National University) for their encouragement in my doctoral career.

Last but not the least, I would like to thank my family: my parents Soonki Jhang and Donghee Jo, for their endless support throughout my life. My wife has been supportive and inspires me in all aspects of my life ever since we met. I deeply appreciate her patience and sacrifice over the course of my doctoral study.

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## 1. INTRODUCTION

Explaining the cross section of stock returns is one of the most important topics in asset pricing, and the value premium anomaly is a key issue in this enterprise. There have been many theoretical and empirical attempts to account for the cause and nature of this phenomenon. Recently the role of cash flow risk has been emphasized in the literature.<sup>1</sup> Several papers have tried to link cash flow risk and cash flow duration to cross-sectional return variation. For instance, Campbell and Vuolteenaho (2004), Bansal, Dittmar, and Lundblad (2005), Kiku (2007), Hansen, Heaton, and Li (2008), Zhang (2005), Lettau and Wachter (2007), Da (2009), Santos and Veronesi (2010), and Choi, Johnson, Kim, and Nam (2013) developed structural models that directly link cash flow risk or cash flow duration with book-to-market and expected stock returns to this end.

When a prototypical asset pricing model produces a cross-sectional variation associated with cash flows, one puzzling feature arises. Growth (value) stocks have longer (shorter) durations, and therefore, they have a higher risk premium in light of discount risk, contrary to the empirical evidence. Thus, economic models that explain the time-series properties of asset prices have difficulty in matching their cross sectional variations and vice versa. Little attention was paid to this issue until recently. Lettau and Wachter (2007, 2011) state that this problem can disappear if the time-varying price of discount-rate risk is uncorrelated with aggregate dividends or consumption in their reduced-form model. However, significant empirical evidence

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<sup>1</sup>Cash flow risk is usually defined by the covariance between a firm's cash flow and the aggregate cash flow. See Abel (1999), Bansal and Yaron (2004), Da (2009) and others for theoretical aspects. For empirical studies see Bansal, Dittmar, and Lundblad (2005), Santos and Veronesi (2006), Yang (2007), Cohen, Polk, and Vuolteenaho (2009), Campbell, Polk, and Vuolteenaho (2010) and many others.

exists that time-varying equity risk premia are countercyclical and closely associated with aggregate consumption or dividends. Thus, this finding needs an economic justification to be compatible with macroeconomic and financial data. Further, Santos and Veronesi (2010) show that, in equilibrium, when the stochastic discount factor is allowed to generate time-varying risk premia correlated with aggregate cash flows, and to account for the aggregate moments of macroeconomic and stock market variables, a value premium can prevail only when aggregate cash flows are counterfactually volatile. Thus, even when firms' cash flows are correctly specified, a growth premium arises, and then a cash flow puzzle appears.<sup>2</sup>

In this paper, we tackle this issue in an exchange economy setting by investigating the effect of idiosyncratic cash flow fluctuations on the cross-section of stock returns. The main departure of our paper is to incorporate belief differences of investors into cash flow dynamics of individual firms, with the following features.

First, we model both aggregate and individual cash flow processes in a consistent way. That is, the aggregate cash flow process is modeled exogenously and it is impacted by aggregate risk only. On the other hand, an individual firm's cash flow process is subject to idiosyncratic risk in addition to aggregate risk. The dynamics of the cash flow processes (including idiosyncratic risk exposure) implies that our model is constructed such that in the aggregate, idiosyncratic cash flow risk cancels out. Second, we introduce investors' heterogeneous beliefs into cash flow processes. The key assumption regarding investors' belief heterogeneity is that investors have different opinions on the long-run mean of the share process with respect to firm- or asset-specific risk.<sup>3</sup> This differs from most of the heterogeneous beliefs literature

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<sup>2</sup>A notable exception is Choi, Johnson, Kim, and Nam (2013). Their model is directly based on firms' dividend policies and focuses on timings of dividend policy by the endogenization of paying and non-paying regimes to reconcile this issue. For related empirical studies, see Da and Warachka (2009) and Dechow, Sloan, and Soliman (2004).

<sup>3</sup>This assumption is similar to Basak (2000) where investors have different beliefs about the cash

where investors update their perceptions of the drift of underlying processes through aggregate risk. Since the market is equipped with a sufficient number of assets to make the asset market complete, idiosyncratic cash flow risk is priced in equilibrium through belief differences.

One of our main theoretical findings is that individual expected stock returns are *positively* affected by idiosyncratic cash flow risk through belief differences.<sup>4</sup> The higher the belief difference, the stronger the effect of idiosyncratic cash flow risks on equilibrium stock returns. Individual equilibrium returns are also positively affected by the cash flow share ratio (long-run mean of the share divided by the contemporaneous share) and negatively by the habit ratio and the interaction between the share ratio and the habit ratio.<sup>5</sup> Cross-sectional return variations result from differences in those variables in addition to idiosyncratic cash flow risk via belief differences. It turns out that value stocks have higher values in the share ratio and belief differences, and this is also consistent with our empirical analysis. Thus, the theory connects these firm characteristics and related investor behavior to the cross section of stock returns. Furthermore, our quantitative study reveals that the cross-sectional return variation is largely attributed to the pricing of idiosyncratic cash flow risk in equilibrium. Specifically, our simulation results state that a growth premium arises with a model where idiosyncratic cash flow risk is ignored, due to the absence of belief differences. This basically replicates Santos and Veronesi (2010). Our results imply

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flow growth rate through non-fundamental risk.

<sup>4</sup>Recently Babenko, Boguth, and Tserlukevich (2013) show that the idiosyncratic cash flow risk negatively affects equilibrium stock return. High idiosyncratic cash flow risk increases the profit of a firm, which in turn increases the firm size. When the firm size increases, the price of risk, measured by CAPM beta, decreases so that the expected excess return decreases. However, as Table 4.4 shows, firms with higher idiosyncratic cash flow risk happen to be value firms. Also Babenko, Boguth, and Tserlukevich (2013) do not take into account the tension between the discount-rate component and cash flow component in generating the value premium, not to mention that idiosyncratic cash flow risk in their model is not priced in equilibrium at all.

<sup>5</sup>cash flow share is defined as the ratio of firm cash flow to the market cash flow. Following Menzley, Santos, and Veronesi (2004), we use the share process to represent individual cash flows.

that sorting stocks based on price-to-fundamental endogenously picks up stocks with higher (idiosyncratic) cash flow risk and *higher degrees of belief differences* in the cross-section so that the value premium arises.

In this light, the main contribution of our paper is to show that idiosyncratic cash flow risk in conjunction with investors' heterogeneous beliefs can explain the cross section of stock returns and the related cash flow dynamics. Another contribution of the paper is to shed light on the characteristics of value and growth stocks. For our quantitative analysis, we estimate individual cash flow share processes by merging data sets involving stock prices, firm characteristics, and analyst forecasts. From our empirical results, value stocks tend to have the lower long-run mean of the share, the higher share ratio of the long-run mean to the current share, and the slower mean reversion of the share than growth stocks. Lower long-run mean of the share can be interpreted that value firms have lower growth potential compared to growth firms. Plus, higher share ratios of value stocks imply that value firms, despite their lower long-run mean of the share, have even a lower current share, implying that the value firms currently suffer from lower profitability.<sup>6</sup> A slower mean reversion of the share process also indicates that value firms may grow more slowly. In addition, value stocks have higher idiosyncratic volatilities of the cash flow share, and the higher degrees of belief differences. Interestingly enough, aggregate cash flow volatilities of the two types of equities do not differ, reinforcing our argument on the importance of idiosyncratic cash flows. In sum, a value stock has a lower growth potential, even a lower current status with a slow speed of convergence to its long-run mean, yet its idiosyncratic fluctuation is higher, and presumably related, market participants have more mixed opinions. The value stock may be a good deal given its longer distance

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<sup>6</sup>The relation between book to market ratio and share ratio is consistent with Avramov, Cederburg, and Hore (2012) and Chen (2013).

between the potential and the current cash flow share, but it comes with more risky individual cash flows and disperse views.

The importance of the pricing of idiosyncratic cash flow risk via belief differences sheds light on the challenge that existing asset pricing models such as Menzley, Santos, and Veronesi (2004), Lettau and Wachter (2007), Santos and Veronesi (2010), and Lettau and Wachter (2011) face. In particular, Lettau and Wachter (2007) and Santos and Veronesi (2010) show that if there is a negative correlation between shocks to aggregate cash flows and shocks to the stochastic discount factor as in the model of Campbell and Cochrane (1999), then the growth premium will prevail in the cross-section that is opposite to the data.<sup>7</sup> Lettau and Wachter (2007) assume that two shocks above have zero correlation. With this, the state variable that derives the stochastic discount factor can suppress the effect of discount-rate risk that is pronounced in previous asset pricing models so that the value premium arises. In our model, while the aggregate shock to the stochastic discount factor is negatively correlated with the shock to aggregate cash flows, investor belief difference related to idiosyncratic shocks also show up in the stochastic discount factor, and they are uncorrelated with shocks to aggregate cash flows. The idiosyncratic cash flow risk in equilibrium via belief differences can suppress the effect of the discount-rate risk in the cross-section so that the value premium arises. Therefore our model provides one equilibrium justification for the finding of Lettau and Wachter (2007) as well as for the counterfactual magnification of cash flow risk in the cross-section of Santos and Veronesi (2010).

In our study, cash flow duration can be clearly defined. The price elasticity with respect to the cash flow represents the inverse of the price elasticity with respect to

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<sup>7</sup>The negative correlation between the two shocks helps explain the equity premium in the aggregate.



the discount rate since the return with cash flow component and the return with discount-rate component moves opposite in the cross-section. Quantitative study shows that value firms have lower cash flow durations than growth firms, which is consistent with the explanations in Lettau and Wachter (2007), Da (2009), and Santos and Veronesi (2010). By generating the value premium, our study confirms the downward sloping equity term-structure in Lettau and Wachter (2011) and van Binsbergen, Brandt, and Koijen (2012).

Finally, our study is related to the heterogeneous beliefs literature of with the following extensions. First, we impose that investors' beliefs work through idiosyncratic risk, unlike the existing work on belief differences that focuses on aggregate risk. Second, our study is related to the literature studying the risk premium relation from belief disagreement. There is conflicting evidence that investors' belief differences lead to either a positive or a negative risk premium.<sup>8</sup> Our equilibrium result shows that a positive risk premium exists, both in time series and cross section, and is closely linked to the value premium anomaly.

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<sup>8</sup>Positive risk premium has been shown in Varian (1985), Varian (1989), Abel (1989), David (2008), Qu, Starks, and Yan (2004), Carlin, Longstaff, and Matoba (2013) and many others. Negative premium has been shown in Miller (1977), Harrison and Kreps (1978), Diether, Malloy, and Scherbina (2005), Chen, Hong, and Stein (2002), Goetzmann and Massa (2005), Johnson (2004), Park (2005), Zhang (2006), and others. Anderson, Ghysels, and Juergens (2005) find both negative and positive risk premium depending on the frequency of measuring belief dispersion.

## 2. THE ECONOMY

### 2.1 Cash Flow Modeling

We consider a continuous-time pure-exchange equilibrium model with two trees. For the specification of the tree process, we follow Menzley, Santos, and Veronesi (2004) by taking the share process as exogenous to describe the relative movement of individual cash flow processes in the economy. The share process is assumed to be a mean-reverting process so that any single asset does not dominate the entire market. This stationarity enables us to analyze the cross-section of stock returns in the long-run. It is also flexible and tractable in modeling risks by which individual assets can have exposures in as many firm-specific risks as possible. Gabaix (2009) shows that this share process belongs to the family of linearity-generating processes such that a closed-form solution can be derived. The share,  $s_t$ , is defined as individual cash flow ( $\equiv D_s(t)$ ) divided by aggregate cash flow ( $\equiv D(t)$ ). Without loss of generality, we analyze one firm and the whole market in this paper. The share process is specified as

$$ds_t = \phi_s(\bar{s} - s_t)dt + \tilde{\sigma}(s_t)d\mathbf{B}'_t, \quad (2.1)$$

where

$$\begin{aligned} \tilde{\sigma}(s_t) &\equiv s_t \cdot \sigma(s_t), \\ \sigma(s_t) &\equiv (\sigma_{s,A}(t), \sigma_{s,I}(t)), \\ \sigma_{s,j} &\equiv v_{s,j} - s_t v_{s,j} - (1 - s_t) v_{(1-s),j}, \quad j = A, I \\ d\mathbf{B}_t &\equiv (dB_A(t), dB_I(t)), \end{aligned} \quad (2.2)$$

where  $\bar{s}$  is the long-run mean of the share of the asset under consideration,  $B_A(t)$  and  $B_I(t)$  represent the aggregate Brownian risk and the idiosyncratic Brownian risk respectively, and  $v_{s,j}$  and  $v_{(1-s),j}$  are the diffusion coefficients of individual assets with the share  $s_t$  and the share  $(1-s_t)$ . Appendix A provides details of the diffusion coefficients to justify (2.1) and (2.2), starting from individual dividend processes.

## 2.2 Belief Difference

Now we incorporate heterogeneous beliefs into the share process.<sup>1</sup> Especially we assume that investors have different beliefs about the long-run mean of the share. We first assume that all investors face the same information about underlying cash flow processes, including both aggregate and individual cash flow processes. In other words, there is no informational asymmetry among investors regarding cash flow processes. Second, we assume that investors agree to disagree to each other, and there is no learning among investors. The theoretical foundation for this “agree-to-disagree” assumption can be found in Varian (1985), Harris and Raviv (1993), and Morris (1994).<sup>2</sup>

Most of the existing work on investors’ belief heterogeneity assumes that investors have different beliefs about the drift of underlying economic processes through aggregate risk. Instead, we assume that investors hold their different beliefs through idiosyncratic risk. Specifically, while all investors agree about aggregate cash flow

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<sup>1</sup>The importance of modeling investors’ heterogeneous beliefs was emphasized early by Lintner (1965), Miller (1977) and Harrison and Kreps (1978). A vast literature that studies the impact of economic agents’ different beliefs about underlying fundamental economic processes on equilibrium quantities now exists. Detemple and Murthy (1994) study the effect of belief differences in a production economy. For exchange economies, see Zapatero (1998), Basak (2000), Basak (2005), Buraschi and Jiltsov (2006), Jouini and Napp (2007), Gallmeyer and Hollifield (2008), David (2008), Dumas, Kurshev, and Uppal (2009), Weinbaum (2009) and others. Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2012) study the impact of belief difference about inflation in an exchange economy.

<sup>2</sup>See also Morris (1996) and Acemoglu, Chernozhukov, and Wold (2007) for settings with belief difference with no learning.

dynamics, they can disagree about the long-run mean of the share of the individual cash flow process through idiosyncratic risk. Our motivation for this assumption is that the information about the aggregate economy is mostly publicly available to all investors. There is, to some degree, a consensus about overall economic conditions. Thus, as long as investors agree about the impact of aggregate risk on underlying cash flow processes, investors' perceptions about cash flow processes including individual ones should be the same. However, when it comes to individual firm information, there can be disagreement about the long-run mean of individual cash flow process that is impacted by firm-specific information. This happens with various reasons even when all the investors share the common information set. For instance, it is possible that investors disagree to each other in interpreting individual firm-specific information due to their different educational backgrounds, different cultural views, or even different cognitive capabilities.<sup>3</sup>

By including the belief difference into the share process, we rewrite the share process as an investor's perceived version as follows.

$$\frac{ds_t}{s_t} = \phi_s \left( \frac{\bar{s}^{(k)}}{s_t} - 1 \right) dt + \sigma_{s,A}(t) dB_A(t) + \sigma_{s,I}(t) dB_I^{(k)}(t), \quad (2.3)$$

where  $k = 1, 2$  refers to the individual investors. Optimal filtering theory<sup>4</sup> implies

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<sup>3</sup>This assumption shares a common feature with rational inattention theory. Rational inattention theory suggests that economic agents process important information first. And if they still have information processing capacity, then they process the remaining information. This assumption comes from the common notion that agents' informational capacity is scarce resource. If we view aggregate risk as the important information and firm-specific risk as residual information, our assumption can be thought of as a special form of rational inattention. For instance, all investors process the aggregate information first in the same manner so that they all agree to how aggregate risk affects underlying economic processes. After that, they process the remaining individual firm-specific information but they can differ in their information processing since they might differ in their remaining resources for individual information processing. See Sims (2003), Sims (2006), and Xiong and Peng (2006) for example.

<sup>4</sup>Liptser and Shiryaev (2001).

that the innovation process  $B_I(t)$  is given by

$$dB_I^{(k)}(t) \equiv \eta_t^{(k)} dt + dB_I(t), \quad (2.4)$$

where  $\eta_t^{(k)} \equiv \frac{\phi_s(\bar{s} - \bar{s}^{(k)})}{\sigma_{s,I}(t)s_t}$ . Note that  $\eta_t^{(k)}$  measures the difference between the true long-run mean of the share and  $k$ -th investor's perceived long-run mean of the share.

The aggregate cash flow process is given by

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_{D,A} dB_A(t). \quad (2.5)$$

Equation (2.5) implies that in the aggregate, the idiosyncratic risks are diversified away. According to the definition of the share process  $s_t$ , an individual cash flow process is defined as the product of the share process and the aggregate dividend. Hence, by applying Ito's lemma to the product of  $s_t$  and  $D_t$ , we have an individually-perceived cash flow process

$$\frac{dD_s(t)}{D_s(t)} = \mu_{D_s}^{(k)}(t)dt + \sigma_{D_s,A}(t)dB_A(t) + \sigma_{D_s,I}(t)dB_I^{(k)}(t) \quad k = 1, 2, \quad (2.6)$$

where

$$\begin{aligned} \mu_{D_s}^{(k)}(t) &\equiv \mu_D + \phi_s \left( \frac{\bar{s}^{(k)}}{s_t} - 1 \right) + \theta_s^{CF} - s_t \theta_s^{CF} - (1 - s_t) \theta_{(1-s)}^{CF}, \\ \sigma_{D_s,A}(t) &\equiv \sigma_{D,A} + \sigma_{s,A}(t), \\ \sigma_{D_s,I}(t) &\equiv \sigma_{s,I}(t), \end{aligned} \quad (2.7)$$

where  $\theta_s^{CF} \equiv v_{s,A} \cdot \sigma_{DA}$ .  $\theta_s^{CF}$  is the unconditional covariance between the share process and the aggregate cash flow process. We define  $\theta_s^{CF}$  as a fundamental cash flow risk parameter following Menzley, Santos, and Veronesi (2004).  $\theta_s^{CF}$  plays an important

role in quantitative study later because it enables us to estimate individual aggregate cash flow parameter  $v_{s,A}$ .<sup>5</sup>

### 2.3 Securities

In our economy, there are three risky assets and one riskless asset. The three risky assets are the market portfolio, an asset with the share process  $s_t$  and an asset with the share process  $1 - s_t$  respectively. Subscripts  $s$  and  $1 - s$  refer to the second and the third assets just defined. Though we have a total of three risky assets, we need to study only two assets due to market clearing. We focus on the assets  $s$  and the market. The price process of the market portfolio is given by

$$\frac{dP_t + D_t}{P_t} = \mu_P(t)dt + \sigma_{P,A}(t)dB_A(t). \quad (2.8)$$

Accordingly, the perceived price of the asset  $s$ , denoted as  $P_s$  is given by

$$\frac{dP_s(t) + D_s(t)}{P_s(t)} = \mu_{P_s}^{(k)}(t)dt + \sigma_{P_s,A}(t)dB_A(t) + \sigma_{P_s,I}(t)dB_I^{(k)}(t) \quad \text{for } k = 1, 2, \quad (2.9)$$

where

$$\mu_{P_s}^{(k)}(t) \equiv \mu_{P_s}(t) - \frac{\sigma_{P_s,I}(t)\phi_s(\bar{s} - \bar{s}^{(k)})}{\sigma_{s,I}(t)s(t)}. \quad (2.10)$$

#### 2.3.1 Investor Preference

Investor preferences are represented by an external habit formation with a constant relative risk aversion utility function. Risk aversion parameters are set to be

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<sup>5</sup>Identification of individual cash flow risk parameters such as  $v_{s,A}$ ,  $v_{s,I}$ ,  $v_{(1-s),A}$  and  $v_{(1-s),I}$  is explained in the appendix, and the roles of the parameters are discussed in the section of quantitative results.

the same across investors for simplicity. Investor  $k$ 's utility function is given by

$$u(c_k(t)) = \frac{1}{1-\gamma} \left( \frac{c_k(t)}{X(t)} \right)^{1-\gamma}, \quad k = 1, 2. \quad (2.11)$$

where  $X(t)$  represents a ratio habit as in Abel (1989). The habit process is defined following Constantinides (1990), Detemple and Zapatero (1991), and Santos and Veronesi (2010);

$$X_t \equiv \delta \int_0^t e^{-\delta(t-\tau)} D_\tau d\tau. \quad (2.12)$$

## 2.4 Equilibrium

As mentioned earlier, we consider two risky assets and one riskless asset in our economy.  $r_t$  is the rate of return for the riskless asset. Turning to the consumption-portfolio problem of the individual investor, investor  $k$ 's wealth  $W^{(k)}(t)$  evolves as

$$\begin{aligned} dW_t^{(k)} = & \left[ r_t W_t^{(k)} - c_t^{(k)} + \pi_M^{(k)}(t) \mu_P(t) + \pi_s^{(k)}(t) \mu_{P_s}^{(k)}(t) \right] dt \\ & + \left[ \pi_M^{(k)}(t) \sigma_{P,A}(t) + \pi_s^{(k)}(t) \sigma_{P_s,A}(t) \right] dB_A(t) \\ & + \left[ \pi_s^{(k)}(t) \sigma_{P_s,I}(t) \right] dB_I^{(k)}(t), \end{aligned} \quad (2.13)$$

where  $c_t^{(k)}$  is the consumption of the  $k$ -th investor,  $\pi_M^{(k)}$  and  $\pi_s^{(k)}$  are the  $k$ -th investor's risky investments in the market portfolio and an asset that corresponds to the share process,  $s_t$ , respectively. The riskless investment is defined as  $b_k(t) \equiv W^{(k)}(t) - \pi_M^{(k)}(t) - \pi_s^{(k)}(t)$ . Following Dybvig and Huang (1988), we impose a non-negativity condition on the wealth process in order to rule out arbitrage strategies.

We now specify state price densities across investors as follows.

$$d\xi_t^{(k)} = -\xi_t^{(k)} \left[ r_t dt + \theta_A(t) dB_A(t) + \theta_I^{(k)}(t) dB_I^{(k)}(t) \right] \quad \text{for } k = 1, 2, \quad (2.14)$$

where  $\theta_A$  is the market price of aggregate risk and the  $\theta_I^{(k)}$ s are the perceived market price of idiosyncratic risk for investor  $k$ . Market prices of risks are<sup>6</sup>

$$\begin{aligned}\theta_A(t) &\equiv \frac{\mu_P(t) - r_t}{\sigma_{P,A}}, \\ \theta_I^{(k)}(t) &\equiv \left[ -\frac{\sigma_{P_s,A}}{\sigma_{P_s,I}} \theta_A(t) + \frac{1}{\sigma_{P_s,I}} (\mu_{P_s} - r) - \eta_t^{(k)} \right].\end{aligned}\tag{2.15}$$

Thus the following link exists between the two idiosyncratic market prices of risks:

$$\theta_I^{(1)}(t) - \theta_I^{(2)}(t) = \eta_t^{(2)} - \eta_t^{(1)} = \bar{\eta}_t.\tag{2.16}$$

For simplicity, we assume the second investor is always the more optimistic investor in the sense that  $\bar{s}^{(2)}$  is bigger than  $\bar{s}^{(1)}$ . This implies that the belief difference term is negative;

$$\bar{\eta}_t \equiv \eta_t^{(2)} - \eta_t^{(1)} = \frac{\phi_s(\bar{s}^{(1)} - \bar{s}^{(2)})}{\sigma_{s,I}(t)s_t} < 0.$$

Investors are assumed to be infinitely lived. In our economy the market is complete. Thus we can formulate an individual optimization problem using martingale methods as follows.

$$\begin{aligned}\max_{c_k} E^{(k)} &\left[ \int_0^\infty u_k(c_k(t)) dt \right] \\ \text{subject to} & \\ E^{(k)} &\left[ \int_0^\infty \xi^{(k)}(t) c_k(t) dt \right] \leq W^{(k)}(0) \equiv w_k P(0),\end{aligned}\tag{2.17}$$

where  $P(t)$  is the total wealth held by both investors at time  $t$  since it is the value of the market portfolio. Also note that  $W^{(1)}(t) + W^{(2)}(t)$  is the total wealth in the economy such that it equals  $P(t)$ .  $w_k$  is the initial fraction of wealth held by investor

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<sup>6</sup>Derivations of the market prices of risks are in the appendix.



$k$  of the market portfolio. From the maximization problem in (2.17), the optimality condition for investor  $k$ 's consumption is given by

$$\begin{aligned} c_k(t) &= I_k \left( \frac{\xi^{(k)}(t)}{\lambda_k} \right) \\ &= \left( \frac{1}{X_t} \right)^{\frac{1-\gamma}{\gamma}} \left[ \frac{\xi^{(k)}(t)}{\lambda_k} \right]^{-\frac{1}{\gamma}}, \end{aligned} \quad (2.18)$$

where  $1/\lambda_k$  is the Lagrange multiplier for investor  $k$ 's optimal consumption-portfolio choice problem, and  $I_k(\cdot)$  is the inverse of investor  $k$ 's utility function. From the static budget constraint of investor  $k$ 's problem, we have

$$\lambda_k = \left[ \frac{E^{(k)} \left[ \int_0^\infty \left\{ \xi^{(k)}(t) X_t \right\}^{\frac{\gamma-1}{\gamma}} dt \right]}{w_k P_M(0)} \right]^{-\gamma}. \quad (2.19)$$

Equilibrium in our economy is then defined as follows.

**Definition 1.** *Given preferences, endowments, and beliefs structures, an equilibrium in this economy is a collection of allocations  $\left( \hat{c}_k, \hat{\pi}_M^{(k)}, \hat{\pi}_s^{(k)}, \hat{b}_k \right)_{k=1,2}$  and a supporting price system*

*$\left( r, \mu_P, \mu_{P_s}^{(k)}, \sigma_P, \sigma_{P_s} \right)$  such that  $\left( \hat{c}_k, \hat{\pi}_M^{(k)}, \hat{\pi}_s^{(k)}, \hat{b}_k \right)$  optimally solves investor  $k$ 's consumption-portfolio choice problem given his/her perceived price processes, and security prices*

are consistent across investors, and all markets clear for  $t \in [0, T]$ :

$$\begin{aligned}
\sum_{k=1}^2 \hat{c}_k(t) &= D(t), \\
\sum_{k=1}^2 \hat{\pi}_M^{(k)}(t) &= 1, \\
\sum_{k=1}^2 \hat{\pi}_s^{(k)}(t) &= s(t), \\
\sum_{k=1}^2 \hat{b}_k(t) &= 0.
\end{aligned} \tag{2.20}$$

In order to derive the equilibrium prices, we must find two stochastic discount factors that clear the consumption goods market:

$$\hat{c}_1(\xi^{(1)}(t)/\lambda_1, t) + \hat{c}_2(\xi^{(2)}(t)/\lambda_2, t) = D(t). \tag{2.21}$$

For computational purpose, we define the stochastic weighting process  $\lambda_t$  as follows:

$$\lambda_t \equiv \frac{\lambda_1 \xi_t^{(2)}}{\lambda_2 \xi_t^{(1)}}, \tag{2.22}$$

where  $\lambda_0 = \frac{\lambda_1}{\lambda_2}$ , since  $\xi^{(k)}(0) = 1$  for  $k = 1, 2$ . As indicated in Gallmeyer and Hollifield (2008) and Bhamra and Uppal (2010),  $\lambda_t$  provides the information about the differences in the investors' opportunity sets given heterogeneous beliefs. Note that the  $\lambda_t$  process is a stochastic weight in the representative investor's utility function for computing the equilibrium as follows.<sup>7</sup>

$$U(C, \lambda) = \max_{c_1 + c_2 \leq D} \frac{\lambda_t (c_1/X)^{1-\gamma}}{\lambda_1 (1-\gamma)} + \frac{1 (c_2/X)^{1-\gamma}}{\lambda_2 (1-\gamma)}, \tag{2.23}$$

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<sup>7</sup>This method goes back to Cuoco and He (1994) and has been applied in many equilibrium studies. See Basak and Cuoco (1998), Basak (2000), Detemple and Serrat (2003), Basak and Gallmeyer (2003), Gallmeyer and Hollifield (2008) and many others.

where  $C \equiv D$ . By applying Ito lemma to  $\lambda_t$ , we obtain the diffusion process of  $\lambda(t)$  as

$$\frac{d\lambda_t}{\lambda_t} = \bar{\eta}_t dB_I^{(2)}(t). \quad (2.24)$$

The process of  $\lambda_t$  is fully described by the disagreements,  $\bar{\eta}_t \equiv [\eta_t^{(2)} - \eta_t^{(1)}]$ , and the idiosyncratic Brownian risk  $B_I^{(2)}(t)$  perceived by the optimistic investor. Using this stochastic weight process, we can write the consumption goods clearing condition as

$$\hat{c}_1(\xi^{(2)}(t)/[\lambda_2 \lambda(t)], t) + \hat{c}_2(\xi^{(2)}(t)/\lambda_2, t) = D(t). \quad (2.25)$$

Since the risk aversion coefficients across investors are the same,  $\gamma$ , the stochastic discount factors for each investor can be obtained as follows.

$$\begin{aligned} \frac{\xi_t^{(2)}}{\lambda_2} &= D_t^{-\gamma} \left( \frac{1}{X_t} \right)^{1-\gamma} \left[ 1 + \left( \frac{1}{\lambda_t} \right)^{-(1/\gamma)} \right]^\gamma, \\ \frac{\xi_t^{(1)}}{\lambda_1} &= \frac{\xi_t^{(2)}}{\lambda_2} \frac{1}{\lambda_t}. \end{aligned} \quad (2.26)$$

### 3. THEORETICAL RESULTS

#### 3.1 Equilibrium Behavior of Price and Return

We now derive equilibrium quantities in the economy. Note that  $\xi^{(1)}$  and  $\xi^{(2)}$  are connected through the  $\lambda_t$  process since, by definition,  $\lambda_t$  is the Radon-Nikodym derivative between two investors' perceived probability measures. Therefore it is sufficient to compute the price of an asset with the share,  $s_t$ , using the second investor's state price density since the price computed from  $\xi^{(1)}$  should be the same as the one that is computed with  $\xi^{(2)}$  in equilibrium. The closed-form solution is obtained by using the same method of Menzley, Santos, and Veronesi (2004).<sup>1</sup> Before we state the result, we first define the process of  $H_t \equiv (D_t/X_t)^{(1-\gamma)}$  as

$$dH_t = h_1(\bar{H} - H_t)dt + h_2H_tdB_A(t) \quad (3.1)$$

following Santos and Veronesi (2010) for the tractability of equilibrium computation. The individual equilibrium price-dividend ratio is given below.

**Proposition 1.** *The equilibrium stock price with the share process  $s_t$  is given by*

$$\frac{P_s(t)}{D_s(t)} = \left[ \beta_{0,t} + \beta_{1,t} \left( \frac{\bar{H}}{H_t} \right) + \beta_{2,t} \left( \frac{\bar{s}^{(2)}}{s_t} \right) + \beta_{3,t} \left( \frac{\bar{s}^{(2)}}{s_t} \frac{\bar{H}}{H_t} \right) \right], \quad (3.2)$$

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<sup>1</sup>Alternatively, we can use an affine transform technique such as Duffie, Pan, and Singleton (2000) or Chen and Joslin (2012). However with this method, the final solution can only be obtained via numerical method as we end up a Ricatti equation system with numbers of differential equations. Instead, we follow Menzley, Santos, and Veronesi (2004) to directly solve the problem.

where

$$\begin{aligned}
\beta_{0,t} &\equiv \int_0^\infty \Psi_2(\tau; t) d\tau, \\
\beta_{1,t} &\equiv \int_0^\infty \frac{\Psi_3(\tau; t)}{\bar{H}}(\tau) d\tau, \\
\beta_{2,t} &\equiv \int_0^\infty \frac{\Psi_4(\tau; t)(\tau)}{\bar{s}^{(2)}} d\tau, \\
\beta_{3,t} &\equiv \int_0^\infty \frac{\Psi_1(\tau; t)(\tau)}{\bar{s}^{(2)} \bar{H}} d\tau,
\end{aligned} \tag{3.3}$$

and the formula of  $\Psi_k(\tau; t)$ 's are given in Appendix C. Note that  $\bar{s}/s_t$  and  $\bar{H}/H_t$  are referenced to the share ratio and the habit ratio respectively.

Proof: See Appendix C

The equilibrium price-dividend ratio of a stock with the share  $s_t$  depends on three main variables: the share ratio( $\bar{s}/s_t$ ), the habit ratio( $\bar{H}/H_t$ ), and the interaction between the share ratio and the habit ratio. First note that all  $\beta_{k,t}$ 's are positive.<sup>2</sup> Thus it seems that we have a simple positive relation between equilibrium price-dividend ratio and three main variables. However, this turns out to be not correct. Note that coefficients  $\beta_{k,t}$ 's are time-varying and cross-sectionally different. They are functions of belief difference,  $\bar{\eta}_t$ , the share,  $s_t$ , the long-run mean of the share,  $\bar{s}$ , and other parameters determining firm characteristics. Especially they are non-linear functions of  $s_t$ . Therefore there is a non-linear relation between equilibrium price-dividend ratio and three variables since all the coefficients are simultaneously affected by changes in each variable. Thus it is hard to qualitatively predetermine the effects of the habit ratio and the share ratio on equilibrium price-dividend ratio.<sup>3</sup>

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<sup>2</sup>This can be shown in the proof of the proposition in the appendix. Also the simulation study later confirms that all coefficients are indeed positive.

<sup>3</sup>The expression is similar to the equilibrium price-dividend ratio in Menzley, Santos, and Veronesi (2004) and Santos and Veronesi (2010). In both studies, coefficients are either positive constants or positive linear functions of the share  $s_t$ . Therefore there are simple positive linear

The exact relations can be characterized by a quantitative analysis with properly calibrated parameters. Though details of quantitative analysis will be discussed later, we briefly describe the relation between equilibrium price-dividend ratio and the three main variables. The habit ratio positively affects the equilibrium price-dividend ratio. Note that the variable  $H_t$  is a macro-type variable related to business cycle. The economy is in a good state when  $1/H_t$  is high and vice versa. The effect from  $H_t$  dominantly translates into the interaction term between the habit ratio and the share ratio so that the equilibrium price-dividend ratio is also positively affected. On the other hand, the share ratio *negatively* affects the equilibrium price-dividend ratio. This is contrary to our preliminary intuition. Changes in the share ratio simultaneously affect all the coefficients  $\beta_{k,t}$ 's so that the counter intuitive relation arises. Indeed, the share ratio turns out to be a crucial variable that governs the cash flow risk both in the cross-section and the time-series. We will discuss this later.

Now we turn to the expected excess return of an asset with share  $s_t$ . We can compute closed-form equilibrium expected stock return as follows.

**Proposition 2.** *The equilibrium expected excess return of a stock with the share process  $s_t$  at time  $t$ , denoted as  $E_t[dR_{s,t}]$ , is given by*

$$E_t[dR_{s,t}] = \left[ \frac{D_s(t)}{P_s(t)} \right] \left[ \mu_{s,t}^{A,I} + \mu_{s,t}^I \right], \quad (3.4)$$

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relations between the equilibrium price-dividend ratio and the share ratio, the consumption surplus ratio, and the interaction between those two variables. Thus high share ratio and high consumption surplus ratio leads to high price-dividend ratio.

where

$$\begin{aligned}
\mu_{s,t}^{A,I} &\equiv \beta_{0,t} (\sigma_{D,A} + \sigma_{s,A}(t)) (\sigma_{D,A} - h_2) + \beta_{1,t} (\sigma_{D,A} + \sigma_{s,A}(t) - h_2) (\sigma_{D,A} - h_2) \frac{\bar{H}}{H_t} \\
&\quad + \beta_{2,t} \sigma_{D,A} (\sigma_{D,A} - h_2) \frac{\bar{s}^{(2)}}{s_t} + \beta_{3,t} (\sigma_{D,A} - h_2)^2 \frac{\bar{s}^{(2)}}{s_t} \frac{\bar{H}}{H_t}, \\
\mu_{s,t}^I &\equiv -\frac{1}{2} \sigma_{s,I}(t) \bar{\eta}_t \left( \beta_{0,t} + \beta_{1,t} \frac{\bar{H}}{H_t} \right).
\end{aligned} \tag{3.5}$$

Proof: See Appendix C

Similar to Proposition 1, individual equilibrium expected excess return depends on the three main variables. By carefully grouping components, we can decompose equilibrium expected excess return into two parts. The first one is mixed with both aggregate and idiosyncratic cash flow risks,  $\mu_{s,t}^{A,I}$ , and the other is a pure idiosyncratic cash flow risk part,  $\mu_{s,t}^I$  which depends on the interaction between the idiosyncratic cash flow risk and investors' belief difference. Most importantly  $\mu_{s,t}^I$  is positive since  $-\sigma_{s,I}(t)\bar{\eta}_t$  is positive due to the fact that  $\bar{\eta}_t$  is negative and  $\beta_{0,t}$  and  $\beta_{1,t}$  are positive. This implies that *the idiosyncratic cash flow risk(volatility) positively affects the equilibrium individual expected excess return!*. The reason for this is given as follows. In our model, idiosyncratic cash flow risk is priced in equilibrium through the channel of belief difference. This is captured by the covariance between the idiosyncratic shock to stochastic discount factor process and the idiosyncratic shock to the share process, which is  $\sigma_{s,I}(t)\bar{\eta}_t$ . Due to an equilibrium restriction, this covariance becomes embedded in  $-(1/2)\sigma_{s,I}(t)\bar{\eta}_t$ , which is positive since  $\bar{\eta}_t$  is negative.<sup>4</sup> Thus

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<sup>4</sup>Equilibrium expected excess return is determined by the negative of the multiplication between the diffusion coefficients of the price process and the diffusion coefficient of the state price density, which is the negative of the covariance between two diffusion processes. The diffusion coefficient of an idiosyncratic Brownian risk of the state price density is given by  $\bar{\eta}_t$ . Thus the negative sign on the covariance between the share process and the belief difference process is positive.

idiosyncratic cash flow risk *positively* affects the expected excess return through the channel of belief difference.<sup>5</sup>

As we mentioned in proposition 1, coefficients  $\beta_{k,t}$ 's are either non-linear functions of  $s_t$  or functions of parameters determining firm characteristics. Thus it is hard to predetermine the effect of the main variables on equilibrium expected excess return. Though the quantitative study can reveal the exact relations, we briefly describe the association between equilibrium expected excess return and the main variables as we did before. The habit ratio, as is related to business cycle, negatively affects equilibrium expected excess return. Also the interaction term between the habit ratio and the share ratio negatively affects the equilibrium expected excess return since the aggregate risk regarding the habit ratio dominantly translates into the interaction term. On the other hand, the share ratio positively affects the equilibrium expected excess return since it represents the cash flow risk of an individual asset. We will discuss this later with great detail.

Note that it is also hard to qualitatively determine the effect of idiosyncratic cash flow risk embedded in  $\mu_{s,t}^{A,I}$  due to complexities in coefficients  $\beta_{k,t}$ 's. However quantitative study shows that idiosyncratic cash flow risk is important in  $\mu_{s,t}^{A,I}$  as will be shown in the later section.

Now we turn to the equilibrium quantities of the market portfolio. Note that this is the special case of  $s_t = 1$  and no belief difference in individual equilibrium quantities. Equilibrium price and return of the market portfolio are given as follows.

**Proposition 3.** *By applying  $s_t \equiv 1$  and no belief difference to individual equilibrium*

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<sup>5</sup>In our model, this implies that belief difference induces the positive risk premium regarding the individual equilibrium expected excess return.



price-dividend ratio, we get the approximate equilibrium expected return as follows.

$$E_t[dR_t] = (\sigma_{D,A} - h_2)^2 + \frac{h_2}{h_1} (\sigma_{D,A} - h_2) \frac{D_t}{P_t}. \quad (3.6)$$

Proof: See Appendix C

Since there is no belief difference in the aggregate, aggregate equilibrium quantities have no exposure to idiosyncratic cash flow risks. Aggregate equilibrium return of the market portfolio depends only on aggregate risk parameter( $\sigma_{D,A}$ ) and parameters in the habit process, i.e.,  $h_1$  and  $h_2$ . Aggregate equilibrium price-dividend ratio depends on habit process that governs aggregate economic condition. Equilibrium price is high when the habit ratio  $H_t$  is low since low  $H_t$  implies the economy is in good condition. Thus equilibrium price of the market portfolio is proportional to aggregate economic condition. This is similar to Massari (2012) and Santos and Veronesi (2010), where the aggregate equilibrium price-dividend ratio directly depends on consumption-surplus ratio. When the economy is getting better, consumption-surplus ratio is increasing and investors become less risk averse. Hence equilibrium price increases. Our model shows that when the economy gets better,  $H_t$  becomes low. As a result, the aggregate equilibrium price-dividend ratio increases due to increased demand on risky assets in the economy.<sup>6</sup>

### 3.2 Return Decomposition

We now analyze components of equilibrium expected excess return of a stock with the share,  $s_t$ . Recently many asset pricing studies have investigated the relative contribution of the discount rate risk component and the cash flow risk component to

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<sup>6</sup>Unlike Massari (2012) and Santos and Veronesi (2010), our model does not allow time-varying risk preferences among investors. Thus we can only think of increased demand on risky assets due to better economic condition.

the cross sectional variation of stock returns.<sup>7</sup> Following this fashion, we attempt to decompose equilibrium individual expected excess return into the discount rate risk part and the cash flow risk part. In particular, we first compute the cash flow risk premium and define the discount rate risk premium as the difference between equilibrium expected excess return and the cash flow risk return. Note that decomposing the equilibrium expected excess return in (3.4) and (3.5) into the cash flow risk component and the discount rate risk component is not so obvious. Idiosyncratic cash flow risk part,  $\mu_{s,t}^I$  can easily be thought of as a component of cash flow risk premium. However decomposing  $\mu_{s,t}^{A,I}$  is subtle since aggregate cash flow risk is connected to the discount rate risk. To overcome this issue, we first compute the cash flow risk premium by focusing on changes in individual cash flows. Note that the cash flow risk component in equilibrium expected excess return is directly affected by changes in individual cash flows,  $D_s(t)$ , and indirectly affected by changes in  $s_t$ . Direct cash flow effect can be obtained by investigating the price elasticity with respect to the cash flow  $D_s$ :

$$\frac{\partial P_s(t)/P_s(t)}{\partial D_s(t)/D_s(t)} = \left[ \beta_{0,t} + \beta_{1,t} \left( \frac{\bar{H}}{H_t} \right) \right] \left( \frac{D_s(t)}{P_s(t)} \right). \quad (3.7)$$

The indirect effect that is caused by  $s_t$  is reflected in the share ratio  $\bar{s}/s_t$ . Thus the expected return that is associated with the share ratio

$$\beta_{2,t} \sigma_{D,A} (\sigma_{D,A} - h_2) \left( \frac{\bar{s}^{(2)}}{s_t} \right) \left( \frac{D_s(t)}{P_s(t)} \right)$$

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<sup>7</sup>See Campbell and Vuolteenaho (2004), Campbell, Polk, and Vuolteenaho (2010) and many others for the study of relative importance of the discount rate risk component and the cash flow risk component in explaining the cross sectional return variation.

is regarded as a part of the cash flow risk premium.<sup>8</sup> Putting these together, the cash flow risk component,  $\mu_{s,t}^{CF}$  can be defined as

$$\begin{aligned} \mu_{s,t}^{CF} \equiv & \mu_{s,t}^I + \left[ \beta_0 + \beta_1 \left( \frac{\bar{H}}{H_t} \right) \right] \left( \frac{D_s(t)}{P_s(t)} \right) (\sigma_{D,A} + \sigma_{s,A}(t)) (\sigma_{D,A} - h_2) \\ & + \beta_2 \sigma_{D,A} (\sigma_{D,A} - h_2) \left( \frac{\bar{s}^{(2)}}{s_t} \right) \left( \frac{D_s(t)}{P_s(t)} \right). \end{aligned} \quad (3.8)$$

As a result, the discount rate risk component,  $\mu_{s,t}^{DR}$  is defined as

$$\mu_{s,t}^{DR} \equiv E_t [dR_{s,t}] - \mu_{s,t}^{CF}. \quad (3.9)$$

The following proposition summarizes the above result.

**Proposition 4.** *The equilibrium expected excess return of a stock with the share,  $s_t$ , is decomposed into the discount rate risk return and the cash flow risk return.*

$$E [dR_{s,t}] = \mu_{s,t}^{DR} + \mu_{s,t}^{CF}. \quad (3.10)$$

In summary, when idiosyncratic cash flow risk is priced in equilibrium through the channel of investors' belief differences, it *positively* affects individual equilibrium expected excess return. However since all the coefficients in equilibrium price-dividend ratio and equilibrium expected excess return are non-linear functions of the share  $s_t$  as well as functions of parameters determining firm characteristics, we cannot pre-determine the effects of the main variables. In order to investigate the effects of the main variables on equilibrium expected excess return both in the cross-section and in the time-series, the quantitative study is inevitable.

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<sup>8</sup>The linkage between the share ratio and the cash flow risk is discussed in depth in the section of quantitative study.

## 4. QUANTITATIVE ANALYSIS

### 4.1 Data

To investigate quantitative aspects of our model, we need to construct cash flows of individual assets, measures of investors' belief differences, and corresponding individual assets' returns data. We begin with investors' belief differences. We use earnings forecasts (EPS forecasts) from Institutional Brokers' Estimate System (I/B/E/S) to extract investors belief differences. Especially we use monthly EPS forecasts with most recent forecasts from I/B/E/S STATSUM data set. At every month, we find latest available analysts' forecasts. If there are missing values in EPS forecasts, we replace them with previously available forecast values. At the end of each forecast period, we take the standard deviation of EPS forecasts. If there are more than one report on the standard deviation, we take the average of them. We then compute the coefficient of variation of EPS forecasts, i.e., the standard deviation of EPS forecasts divided by mean EPS forecasts.<sup>1</sup> We use this quantity as a belief difference measure,  $-\bar{\eta}_t$ .<sup>2</sup> This method of constructing a belief difference measure is similar to Diether, Malloy, and Scherbina (2005) and Yu (2011).<sup>3</sup> Note that at every point in time, the belief difference measure is determined prior to actual report date of an asset's return. More specifically, at each month, we have a stock's belief difference measure

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<sup>1</sup>We take the coefficient of variation of EPS forecasts to get rid of size effect embedded in standard deviation since the bigger the market size of a stock, the larger the standard deviation. Also note that this measure is invariant in its direction in portfolio deciles sorted by book-to-market when we replace the mean EPS forecasts with the book-equity price. This invariance has been explored in Diether, Malloy, and Scherbina (2005).

<sup>2</sup>Note that the quantity,  $\bar{\eta}$ , is defined as the difference between pessimistic investor's long-run mean of a share and optimistic investors' long-run mean of a share. Thus  $\bar{\eta}$  is negative by definition. Since we define investors' belief difference as the standard deviation of EPS forecasts, it must be a positive quantity. Thus we put minus in front of  $\bar{\eta}$  so that we have a positive belief difference measure.

<sup>3</sup>This method is also similar to using confidence interval as a measure of belief difference. See David (2008).

whose value is determined in the previous month.

For stock price and return data, we use Center for Research in Security Prices (CRSP) and COMPUSTAT merged data. Depending on our purpose, we use either CRSP-COMPUSTAT merged data set or CRSP-COMPUSTAT-I/B/E/S merged data set from January, 1983 to December, 2011. In doing so, we fill missing data values, if any, with the previously available data. One caveat of the CRSP-COMPUSTAT-I/B/E/S merged data set is the coverage of stocks. Due to a low coverage of stocks in I/B/E/S data set, we have about 43% of the whole universe of CRSP stocks.

In order to construct cash flows of individual stocks, we follow Stephens and Weisbach (1998) and Grullon and Michaely (2002) in order to utilize accounting information in COMPUSTAT. We define the cash flow as the sum of dividends and stock repurchases. We construct time-series of cash flows of each stock on a monthly basis. We first compute dividends using returns with and without dividends. The difference of these two gives us dividend yield. Multiplying the dividend yield by the market capitalization of a stock is defined as the dividend. For market capitalization, we take the multiplication between the mid price of a stock within a month and the outstanding number of shares of a stock.<sup>4</sup> For stock repurchases, we first compute the decrease in numbers of outstanding shares of a stock every month by using accounting information in COMPUSTAT. If this decrease is positive (number of shares decreases), then we multiply the decrease in number of shares with the mid-price of a stock in a month. If decreases in outstanding number of shares are negative (numbers of shares increase), then we take zero as a stock repurchase, which is the same as Stephens and Weisbach (1998). Finally we take the moving sum of current and past two months of dividends with share repurchases as the monthly cash flow. This is because at the firm level, there are many months in which cash

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<sup>4</sup>We use the mid point of prices within a month for the purpose of conservatism.

flow data are missing. Moving sum can significantly mitigate the missing value problem. Aggregate cash flow is computed by summing individual cash flows across all individual firms. After constructing individual cash flows, we assign each stock to its Book-to-Market decile using the NYSE breakpoints. Sorting procedure can be found in Kenneth French’s website.

Table 4.1 shows basic summary statistics on the market portfolio and portfolios in value decile. Panel A shows summary statistics of the market portfolio from 1983 to 2011. Panel B and C show basic cross-sectional statistics of average returns, average book-to-market ratio, average price-dividend ratio, and the Sharpe ratio of decile portfolios sorted by book-to-market from CRSP-COMPUSTAT merged data and CRSP-COMPUSTAT-I/B/E/S merged data respectively. Figure 4.1 shows the cross-section of stock returns from CRSP-COMPUSTAT merged data set in panel B of table 4.1. The value premium is clearly pronounced from 1983 to 2011 in both the table and the figure. When stocks are sorted based on book-to-market, stocks with high book-to-market (value stocks) earn about 0.5% extra return per month over stocks with low book-to-market (growth stocks) on average. We have a similar pattern in average returns across stocks that are sorted on price-dividend ratio, though sorting stocks based on price-dividend ratio does not exactly match stocks sorted on book-to-market. For example, stocks with the lowest price-dividend ratio does not exactly correspond to stocks with the highest book-to-market. Data on cash flows will be described later with Table 4.4.

## 4.2 Choice of Parameters

We estimate the parameters of individual cash flow processes using 10 book-to-market portfolios from 1983 to 2011 with CRSP-COMPUSTAT and CRSP-COMPUSTAT-I/B/E/S merged data sets respectively. As is shown in the appendix, individual

Table 4.1: Summary Statistics of the Data – Panel A summarizes basic statistics for the market portfolio from 1983 to 2011 on a monthly basis. Mean return of the market,  $\overline{R_M^e}$ , is the average of excess returns on the market portfolio.  $r_f$  is the riskless rate of return. Return and volatility are expressed in percentage. Panel B and C summarizes key cross sectional moments for book-to-market decile portfolios for CCM (CRSP-COMPUSTAT merged) data and CCIM (CRSP-COMPUSTAT-I/B/E/S merged) data respectively for 1983 to 2011. Returns and volatilities are expressed in percentage. SR is the Sharpe ratio,  $R^e$  is the excess return, and the overline in each variable indicates the sample average.

Panel A: Summary statistics on the market portfolio										
	$\overline{R_M^e}$	$\overline{\sigma_{R_M^e}}$	SR	$\overline{r_f}$	$\overline{\sigma_{r_f}}$					
	0.57	4.57	0.126	0.36	0.22					
Panel B: Statistics on decile portfolio CCM										
	Growth									Value
Portfolio	1	2	3	4	5	6	7	8	9	10
$\overline{R^e}$	0.89	0.99	1.04	0.99	0.96	1.10	1.07	1.06	1.07	1.41
$\overline{B/M}$	0.15	0.309	0.415	0.514	0.614	0.72	0.84	0.99	1.22	2.43
$\overline{P/D}$	159	123	116	100	93	92	85	83	89	92
SR	0.103	0.13	0.139	0.127	0.121	0.162	0.14	0.146	0.123	0.152
Panel C: Statistics on decile portfolio CCIM										
	Growth									Value
Portfolio	1	2	3	4	5	6	7	8	9	10
$\overline{R^e}$	0.953	1.00	1.02	0.97	0.91	1.08	1.1	1.01	1.04	1.53
$\overline{B/M}$	0.167	0.307	0.416	0.513	0.615	0.721	0.841	0.986	1.218	2.08
$\overline{P/D}$	326	289	246	180	237	163	170	211	200	280
SR	0.1139	0.1271	0.1307	0.1193	0.1075	0.1431	0.1356	0.1228	0.1126	0.1477

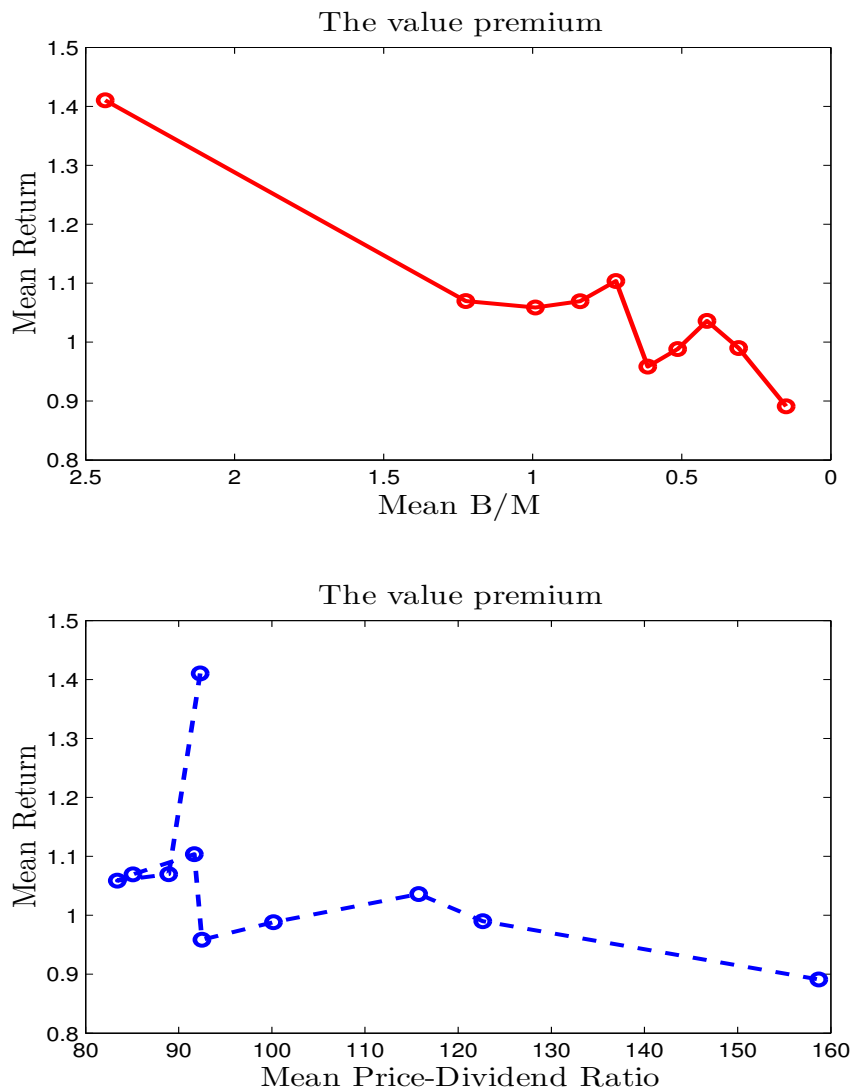


Figure 4.1: Value Premium in the Data – January, 1983 to December, 2011. The data is adopted from CRSP-COMPUSTAT merged data. Stocks are sorted based on both Book-to-Market ratios and Price-to-Dividend ratios whose breakpoints are given in Kenneth French's Data Library.



aggregate cash flow risk parameters,  $v_{s,A}$ ,  $v_{s,I}$ ,  $v_{(1-s),A}$  and  $v_{(1-s),I}$  can be estimated by using restrictions that are imposed on the share process and an individual cash flow process. We describe the details of how to estimate cash flow risk parameters in the later section where the simulation method is explained.

Basic aggregate parameters, i.e.,  $\gamma$ ,  $\delta$ ,  $h_1$ , and  $h_2$ , are calibrated by matching first and second aggregate moments to their data counterparts. Table 4.2 and ?? report the matched moments and the resultant calibrated values of the parameters. Details of calibration method is described in the appendix.

### 4.3 Estimation Results: Characterizing the Cash Flow Risk of Value and Growth Stocks

In this subsection we discuss economic implications of empirical findings on cash flows in value decile. Table 4.4 and 4.5 shows three distinctive features of individual cash flow processes in the data. These features are crucial to understand the economic mechanism of cash flow risks that works behind the value premium. First, the long-run mean of the share is higher in growth firms than value firms. The long-run mean of the share represents a firm's long-run growth since it measures how well a firm has performed in terms of total payout. A firm with a higher long-run mean of the share indicates that it has a relatively higher growth potential.

Second, the estimate of the share ratio,  $\bar{s}/s_t$ , is higher in value firms than growth firms. Table 4.4, 4.5 and figure 4.2 show details of the share ratio properties in value decile. We measure several different estimates of the share ratio using the average of  $\bar{s}/s_t$  over time, quantile values of share ratio, and the average of share ratio in each quantile group. All measures of the share ratio indicate that the share ratio of value stocks is higher than that of growth stocks. Also the time-series of the share ratio of value stocks is, in general, higher than the time-series of the share ratio of

Table 4.2: Calibration – Panel A shows the calibration of aggregate moments using monthly data from 1983 to 2011. Panel B shows the calibration of aggregate moments using quarterly data from 1946 to 2011. Both calibrations use the same aggregate equilibrium equations in matching expected moments to their sample counterparts. Returns and volatilities are expressed in percentage. \* indicates the matched moments.

Panel A: Monthly data from 1983 to 2011		
	Data	Calibration
Average Excess Return*	0.57	0.6
Volatility of Excess Return	4.61	9.8
Sharpe Ratio	0.128	0.061
Average P/D *	93.36	93.36
Average Riskless Rate *	0.036	0.036
Volatility of Riskless Rate	0.22	0.55
Panel B: Quarterly data from 1946 to 2011		
	Data	Calibration
Average Excess Return *	1.9	1.9
Volatility of Excess Return	8.3	12.96
Sharpe Ratio	0.229	0.1466
Average P/D *	93.36	93.36
Average Riskless Rate *	0.035	0.035
Volatility of Riskless Rate	0.22	0.25

Table 4.3: Calibrated Parameters – This table shows calibrated parameters that correspond to Panel A in table 4.2.

$\mu_D$	$\sigma_{D,A}$	$\gamma$	$\delta$	$h_1$	$h_2$
0.02	0.13	3.7	0.9	0.0107	0.08851

Table 4.4: Characteristics of Decile Portfolio – This table shows basic statistics of characteristics of decile portfolios sorted by book-to-market from 1983 to 2011 for CRSP-COMPUSTAT(CCM) and CRSP-COMPUSTAT-I/B/E/S(CCIM) respectively.  $\theta^{CF}$  is an unconditional covariance between the share process and the aggregate cash flow process.  $v_{s,A}$  is pinned down by the relation  $\theta^{CF} = v_{s,A}\sigma_{D,A}$ .  $v_{s,I}$  can be computed from the identification condition imposed on the share process. Details of determining individual cash flow risk parameters are suggested in the appendix. Mean-reverting coefficients  $\phi_s$  are estimated using generalized least square estimation using  $\sigma_{s,A}$  and  $\sigma_{s,I}$ . Coefficient of variation of  $s_t$ , the ratio of the standard deviation of  $s_t$  to the mean of the share  $s_t$  for each portfolio is defined as  $CV(s_t)$ .

Panel A: CCM, 1983 to 2011										
	Growth									Value
Portfolio	1	2	3	4	5	6	7	8	9	10
$\theta^{CF}$	-0.028	-0.025	-0.030	-0.032	-0.028	-0.035	-0.027	-0.031	-0.031	-0.025
$v_{s,A}$	-0.184	-0.166	-0.195	-0.205	-0.181	-0.229	-0.177	-0.204	-0.201	-0.163
$v_{s,I}$	0.205	0.234	0.222	0.214	0.240	0.206	0.276	0.283	0.300	0.458
$\phi_s$	0.068	0.077	0.120	0.078	0.118	0.123	0.123	0.133	0.096	0.039
$\bar{s}$	0.131	0.108	0.080	0.077	0.078	0.064	0.067	0.059	0.051	0.036
$\text{Avg}(\frac{\bar{s}}{s_t})$	1.249	1.214	1.140	1.209	1.224	1.208	1.315	1.290	1.444	2.140
$CV(s_t)$	0.415	0.40	0.31	0.39	0.431	0.40	0.614	0.499	0.713	1.08

Panel B: CCIM, 1983 to 2011										
	Growth									Value
Portfolio	1	2	3	4	5	6	7	8	9	10
$\theta^{CF}$	-0.027	-0.017	-0.029	-0.033	-0.030	-0.031	-0.026	-0.029	-0.028	-0.033
$v_{s,A}$	-0.176	-0.111	-0.186	-0.213	-0.197	-0.201	-0.166	-0.190	-0.184	-0.215
$v_{s,I}$	0.354	0.434	0.437	0.442	0.365	0.451	0.368	0.579	0.657	0.624
$\phi_s$	0.14	0.08	0.041	0.033	0.022	0.038	0.025	0.016	0.02	0.011
$\bar{s}$	0.0039	0.0039	0.0032	0.0031	0.0037	0.0022	0.0026	0.0021	0.0025	0.0017
$\text{Avg}(\frac{\bar{s}}{s_t})$	1.3647	1.5656	2.0512	2.1637	2.4594	2.0390	2.5351	3.2580	3.5944	6.2508
$\text{BD}(\equiv -\bar{\eta})$	0.045	0.053	0.081	0.085	0.102	0.126	0.174	0.160	0.273	0.419

Table 4.5: Share Ratio of Decile Portfolio – This table shows quantile values, average values in each quantile group, and the percentage of share ratios less than 1 of share ratios of decile portfolios sorted by book-to-market from 1983 to 2011 in CRSP-COMPUSTAT(CCM) data set.

CRSP-COMPUSTAT, 1983 to 2011										
	Growth									Value
Portfolio	1	2	3	4	5	6	7	8	9	10
Quantile value	0.821	0.829	0.867	0.867	0.798	0.819	0.900	0.814	0.791	0.817
	1.037	1.016	1.011	1.056	1.079	1.074	1.129	1.075	1.304	1.183
	1.476	1.442	1.228	1.326	1.450	1.373	1.760	1.594	1.794	2.779
	11.107	10.886	8.654	11.020	6.640	6.434	3.912	4.684	5.800	12.115
Average in Quantile	0.660	0.696	0.743	0.694	0.639	0.658	0.614	0.624	0.551	0.566
	0.922	0.909	0.942	0.961	0.960	0.961	1.007	0.952	1.084	0.989
	1.232	1.218	1.109	1.170	1.250	1.192	1.399	1.279	1.54	1.832
	2.184	2.032	1.765	2.012	2.048	2.023	2.239	2.306	2.601	5.172
% of share ratio < 1	45.4	49.1	46.8	42.2	40.8	40.2	36.8	42.0	32.5	37.4

growth stocks as in Figure 4.2. The variation of share ratios in the cross-section has

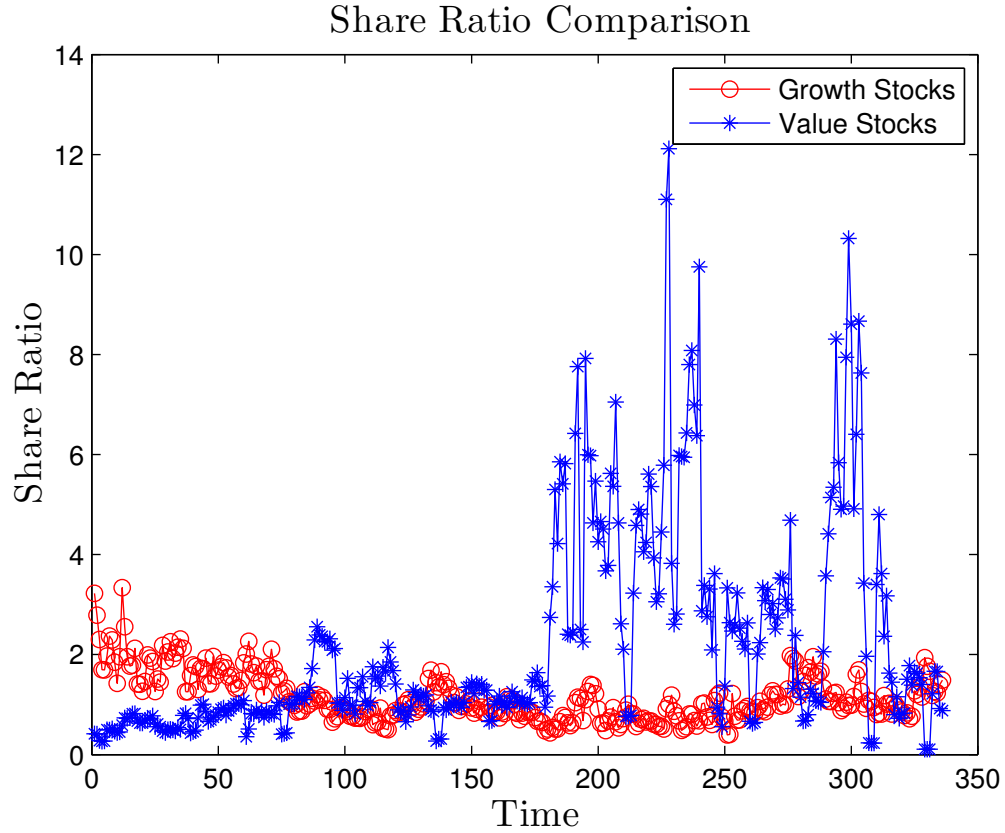


Figure 4.2: Share Ratios in the Data – January, 1984 to December, 2011. The data is adopted from CRSP-COMPUSTAT merged data. Blue starred line represents share ratios of the value stocks. Red circled line represents share ratios of the growth stocks.

important implications about firms' profitabilities and cash flow fluctuations. Higher share ratios in value stocks implies that value stocks' current shares are even lower despite the low value of the long-run mean of shares. This further implies that value firms tend to suffer from lower profitability than growth firms. For instance, a firm that used to be profitable but currently not, pays lot less current dividends than

before so that its present share can be far away from its long-run mean of the share. On the other hand, a firm that is currently profitable is likely to pay dividends that are similar to its long-run mean of the share. This firm (most likely a growth firm) will have a low share ratio even if it has high long-run mean of the share. Besides the profitability, we can infer that higher share ratio is associated with relatively bigger cash flow fluctuations. This inference is derived from the data. The bottom of each panel of Table 4.4 shows the coefficient of variation of the share in value decile. The coefficient of variation of a value firm is higher than that of growth firms. This implies that value firms have larger exposure to current cash flow fluctuations. Higher cash flow fluctuation in value stocks is also captured by higher idiosyncratic cash flow risk,  $v_{s,I}$  in value stocks as aggregate cash flow risks,  $v_{s,A}$  are not so different in value decile.

Third, estimated mean-reverting coefficient is higher in growth firms than in value firms. Note that the mean-reverting coefficient,  $\phi_s$  measures how quickly a firm's share can catch up with its long-run mean. Low value of  $\phi_s$  of value firms implies that, despite low value of long-run mean of the share, value firms have hard time to catch up with its long-run growth potential. Technically speaking, due to a low value of  $\phi_s$ , shares of value firms, compared to growth firms, show larger swing movement in the time-series. Thus value firms are exposed to larger volatility than growth firms.

Related, we can infer an important aspect of consumption risk from belief difference in the data. As the last row of Panel B in Table 4.4 shows, belief difference is positively (negatively) associated with the share ratio (cash flow growth). A bigger volatility of value firms is related to a larger consumption risk. Equilibrium consumption sharing rules are exposed to both aggregate and idiosyncratic risk.<sup>5</sup> Aggregate

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<sup>5</sup>See the appendix for additional equilibrium results.

risk is the same as the risk of the aggregate cash flow process. On the other hand, idiosyncratic consumption risk depends on investors' belief differences. This implies that, in our model, value firms induce larger idiosyncratic consumption risk since they are exposed to higher investors' belief differences through which idiosyncratic cash flow risk translates into equilibrium consumption. Therefore investors will be exposed to larger idiosyncratic consumption risk when they invest in value firms due to higher idiosyncratic cash flow risk.

In summary, empirical features of cash flow data imply that value firms are riskier than growth firms in terms of cash flow risk since they are subject to relatively more fluctuations of current (idiosyncratic) cash flows and higher investors' belief differences.

## 4.4 Simulation Results

### 4.4.1 Simulation Method

In this section, we describe the details of the simulation. We first compute unconditional covariance between the share process and the aggregate cash flow process for 10 book-to-market sorted portfolios. This covariance yields the fundamental aggregate cash flow risk,  $\theta_s^{CF}$ . Dividing this by the diffusion coefficient of the aggregate cash flow process,  $\sigma_{D,A}$ , enables us to pin down individual aggregate cash flow risk parameter,  $v_{s,A}$ .<sup>6</sup> Given this parameter, we proceed to compute the total variability of individual cash flow process that is defined by the multiplication of the share process and the aggregate cash flow process. The total variability of individual cash flow process gives us a restriction by which we can compute individual idiosyncratic cash flow risk parameter,  $v_{s,I}$ . Finally we use the identification condition of the share process to compute  $v_{(1-s),A}$  and  $v_{(1-s),I}$ . When these parameters are obtained, we

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<sup>6</sup>Diffusion coefficient of the aggregate cash flow process,  $\sigma_{D,A}$  is estimated by using the maximum likelihood method.

can estimate the mean-reverting coefficient of the share process using the generalized least square method since  $\sigma_{s,A}$  and  $\sigma_{s,I}$  can be computed using the parametric restrictions imposed on the share process. Table 4.4 shows parameter values across 10 book-to-market portfolios. We simulate 200 firms for 5,000 months and report the result with the latest 2,000 months. For firms, we assign parameters values that are similar to estimated ones in the cross-section. More specifically, we put the first set of values of  $\phi_s$ ,  $v_{s,A}$  and  $v_{s,I}$  in Panel A of Table 4.4 with some fluctuations on first 20 firms (growth firms) and put second set of values of  $\phi_s$ ,  $v_{s,A}$  and  $v_{s,I}$  with some fluctuations on next 20 firms, and so on. For the initial value of the long-run mean of the share, we start from either  $\bar{s} = 0.05$  for each firm or mimicked long-run mean shares from Table 4.4. Since I/B/E/S data is a survey data of analyst forecasters, we suspect there might be an upward bias due to extreme forecasts or low analysts coverage in small firms. Thus we use belief difference parameters between 0.04 to 0.2 that are less volatile than estimated values in the cross-section.

#### 4.4.2 *Idiosyncratic Cash Flow Risk, Belief Difference, and the Return Decomposition*

In the simulation study, we first look at the time-series relation between equilibrium price-dividend ratio (equilibrium expected excess return) and the three main variables. Brief descriptions are suggested in Proposition 1 and 2. Figure 4.3 and 4.4 show the relations with respect to equilibrium price-dividend ratio and equilibrium expected excess return respectively. As mentioned earlier, the habit ratio and the interaction between the habit ratio and the share ratio positively (negatively) affect equilibrium price-dividend ratio (equilibrium expected excess return) in the time-series. The habit ratio represents the fundamental risk in the economy since  $H_t \equiv (D_t/X_t)^{1-\gamma}$  is a macro-type variable indicating the state of the economy. Ac-



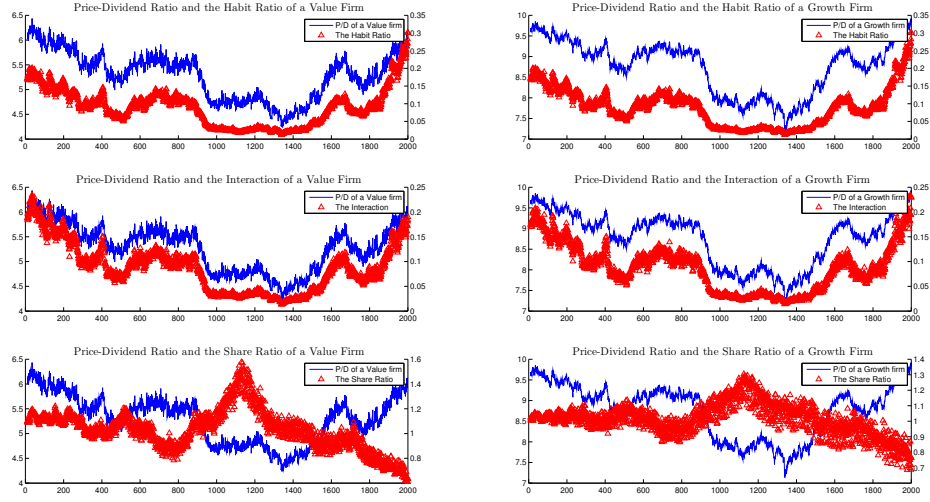


Figure 4.3: The Relation between Equilibrium Price-Dividend Ratio and the Three Main Variables – The model is simulated with  $(\gamma, \delta) = (3.7, 1.9)$  and  $(h_1, h_2) = (0.0107, 0.089)$ . The top two figures show the relation between equilibrium price-dividend ratios and the habit ratio for a value firm and a growth firm respectively. The middle two figures show the relation between equilibrium price-dividend ratios and the interaction term between the habit ratio and the share ratio for a value firm and a growth firm respectively. The bottom two figures show the relation between equilibrium price-dividend ratios and the share ratio for a value firm and a growth firm respectively.

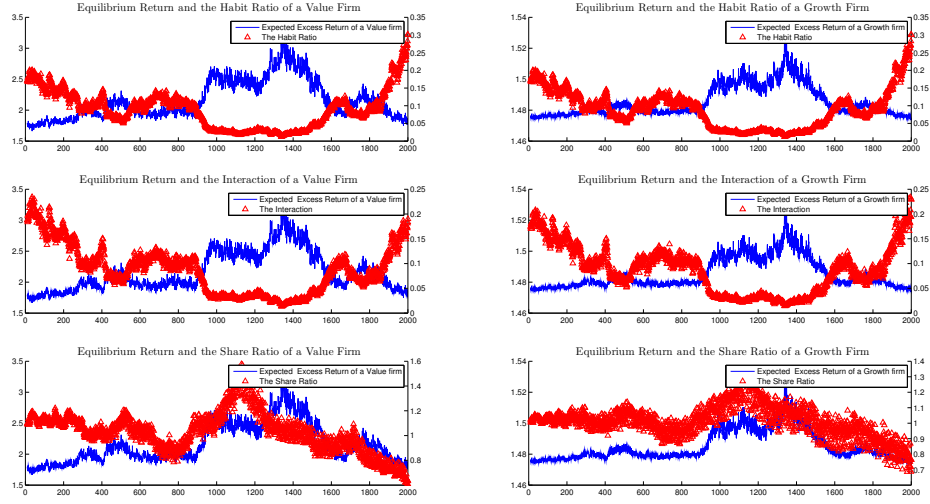


Figure 4.4: The Relation between Equilibrium Expected Excess Return and the Three Main Variables – The model is simulated with  $(\gamma, \delta) = (3.7, 1.9)$  and  $(h_1, h_2) = (0.0107, 0.089)$ . The top two figures show the relation between equilibrium expected excess return and the habit ratio for a value firm and a growth firm respectively. The middle two figures show the relation between equilibrium expected excess return and the interaction term between the habit ratio and the share ratio for a value firm and a growth firm respectively. The bottom two figures show the relation between equilibrium expected excess return and the share ratio for a value firm and a growth firm respectively.

cording to the definition of the variable and the assumption of  $\gamma > 1$ , the economy is slow when  $H_t$  is low and vice versa. For instance, when the economy is in a good state, there will be higher demand on a risky asset so that the price of an asset increases and the resulting expected return decreases.<sup>7</sup> Fundamental risk induced by  $H_t$  dominantly translates into the interaction term. Thus the interaction term also positively (negatively) affects the equilibrium price-dividend ratio (equilibrium expected excess return). Given the fact that coefficients  $\beta_{k,t}$ 's are positive, these two relations are well expected. On the other hand, the effect of the share ratio appears to be counter-intuitive since it negatively (positively) affects equilibrium price-dividend ratio (equilibrium expected excess return). Note that all the coefficients  $\beta_{k,t}$ 's are functions of parameters determining firm cash flow characteristics as well as the share  $s_t$ . This implies that all the coefficients are simultaneously affected by the share ratio as they are time-varying. Thus the relation between equilibrium price-dividend ratio and the share ratio can be highly non-linear so that the seemingly counter intuitive relation arises. However, the interpretation of this result can be straightforward. In the previous section, we discussed empirical features of cash flow data and its economic implications in the cross-section. A firm with high share ratio is exposed to more fluctuations of current cash flows than a firm with low share ratio. As seen in Table 4.6 below, the simulation replicates empirical features of the cash flow processes. Since our model is stationary, we can apply cross-sectional cash flow risk argument to the time-series result.<sup>8</sup> When a firm's share ratio goes up, the firm gets exposed to higher current cash flow risk. Increased cash flow risk leads to

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<sup>7</sup>Note that the habit process in our model does not induce time-varying risk preference since it enters the utility function as the ratio. On the other hand, if habit enters the utility function in the form of difference such as Campbell and Cochrane (1999), then consumption-surplus ratio induces time-varying risk preferences.

<sup>8</sup>Our model does not have non-stationary stochastic processes. Thus the economic reasoning about cash flow risk in the cross-section can be applied to the time-series.

lower equilibrium price-dividend ratio, hence higher equilibrium rate of return. This relation is shown clearly in the bottom panels of Figure 4.3 and 4.4.

Now we investigate cross-sectional properties of the simulation results. Table 4.6 and the first panel of Figure 4.5 show the basic cross-sectional results of our model. Equilibrium expected excess returns sorted by price-dividend ratios show the sizable value premium and resemble the empirical cross-sectional pattern in Figure 4.1. Sharpe ratios are increasing from growth to value firms in value decile, which is also consistent with the empirical evidence in Table 4.1. As the top panel of Figure 4.6 shows, cash flow risk return captures the most of the variation in the cross-section of equilibrium excess returns. Discount discount rate risk return shows the growth premium, although its cross-sectional magnitude is rather small.

Note that the pure idiosyncratic cash flow risk part,  $\mu_{s,t}^I$ , takes a small portion in cash flow risk component as the middle panel of Figure 4.6 shows. However this does not mean that idiosyncratic cash flow effect is minor. As mentioned before,  $\beta_{k,t}$ 's are functions of belief difference,  $\bar{\eta}_t$ , the share,  $s_t$ , the long-run mean of the share,  $\phi_s$ , and other parameters determining firms' cash flow characteristics including idiosyncratic cash flow risk parameter,  $v_{s,I}$ . Thus idiosyncratic cash flow risk can affect  $\mu_{s,t}^{A,I}$  through the beta coefficients via the channel of belief difference.<sup>9</sup> However the true magnitude of the effect of idiosyncratic cash flow risk in  $\mu_{s,t}^{A,I}$  can only be evaluated by comparing our simulation with the simulation of a benchmark model where we do not price idiosyncratic cash flow risk in equilibrium by turning off the channel of investors' belief difference. As the second panel of Figure 4.5 shows, the benchmark model simulation yields the counterfactual growth premium.<sup>10</sup> In the

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<sup>9</sup>It should be pointed out that idiosyncratic cash flow risk,  $v_{s,I}$  is always attached to belief difference,  $\bar{\eta}_t$  in equilibrium. Thus the effect of idiosyncratic cash flow risk is magnified when it is positively associated with belief difference.

<sup>10</sup>The benchmark simulation is carried out using the same cash flow risk parameters in value decile but with the fixed mean-reverting coefficient of the share process at 0.07. This method of

Table 4.6: Model Simulation – This table shows simulation results of the model. 200 firms for 5,000 months were simulated. Individual expected excess returns are generated according to the equation (3.4). Individual assets are sorted into decile portfolios based on simulated price-dividend ratios following (3.2). Other return quantities follow equations (3.8) and (3.9). Sharpe ratios are computed by using expected volatility that can be calculated using (C.34) in the appendix. For the simulation, we assign parameter values for firms following Table 4.4. Average individual aggregate cash flow risk parameters  $v_{sA}$  are from -0.184 to -0.163 and the average individual idiosyncratic cash flow risk parameters  $v_{sI}$  are from 0.205 to 0.458 across decile portfolios. Average mean-reverting coefficients  $\phi_s$  are taken from 0.039 to 0.133 with the order in Table 4.4. Coefficient of variation of the share,  $CV(s_t)$ , is the ratio of the standard deviation of shares to the mean of shares in each portfolio.

Portfolio	Growth										Value
	1	2	3	4	5	6	7	8	9	10	
Average excess return	0.47%	0.48%	0.53%	0.57%	0.59%	0.67%	0.7%	0.74%	0.85%	1.6%	
Average $\ln(P/D)$	8.59	8.16	7.38	7.08	6.82	6.38	6.23	6.10	5.87	5.19	
CF component	0.05%	0.07%	0.13%	0.17%	0.20%	0.29%	0.33%	0.37%	0.50%	1.28%	
ICF component	0.001%	0.002%	0.005%	0.008%	0.011%	0.018%	0.024%	0.031%	0.052%	0.265%	
DR component	0.42%	0.42%	0.41%	0.40%	0.39%	0.37%	0.37%	0.36%	0.35%	0.32%	
Average share ratio	1.022	1.024	1.024	1.026	1.026	1.026	1.034	1.038	1.039	1.077	
Price elasticity w.r.t. CF	0.013	0.014	0.014	0.015	0.016	0.018	0.021	0.027	0.064	0.082	
Sharpe ratio	0.123	0.124	0.126	0.128	0.128	0.132	0.134	0.136	0.143	0.19	
$CV(s_t)$	0.113	0.122	0.12	0.122	0.121	0.118	0.133	0.137	0.14	0.20	

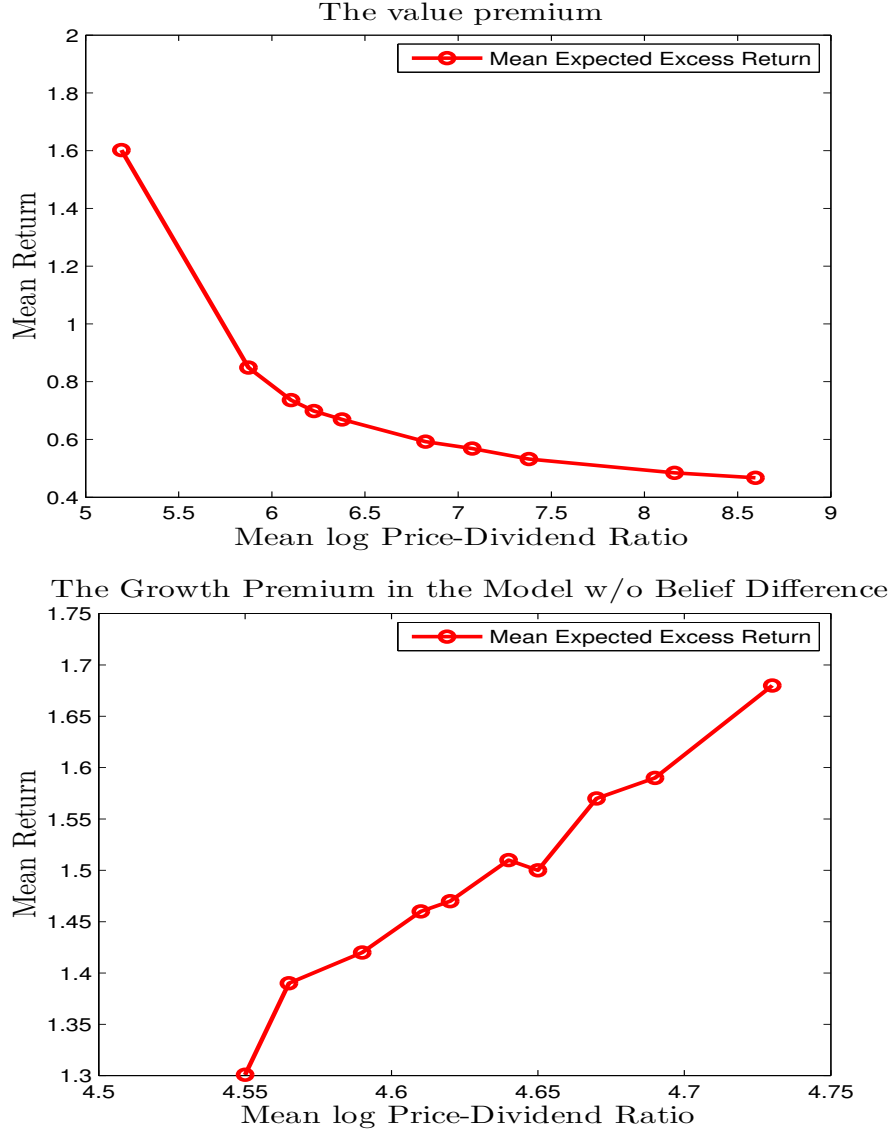


Figure 4.5: Model Comparison with/without Belief Difference – The first figure shows simulated cross-sectional average excess returns sorted by price-dividend ratio from our model. The Second figure is the simulated cross-sectional average excess returns sorted by price-dividend ratio from the benchmark model where investors’ belief differences do not exist. In this case,  $v_{s,I}$  does not exist in equilibrium, neither  $\sigma_{s,A}$  does. Benchmark result roughly corresponds to the cross-sectional return simulation of Santos and Veronesi (2010) in the sense that we fix the mean-reverting coefficient,  $\phi_s$ , for all assets at 0.09. The simulation method is the same as the description in Table 4.6. Mean returns are expressed in percentage.

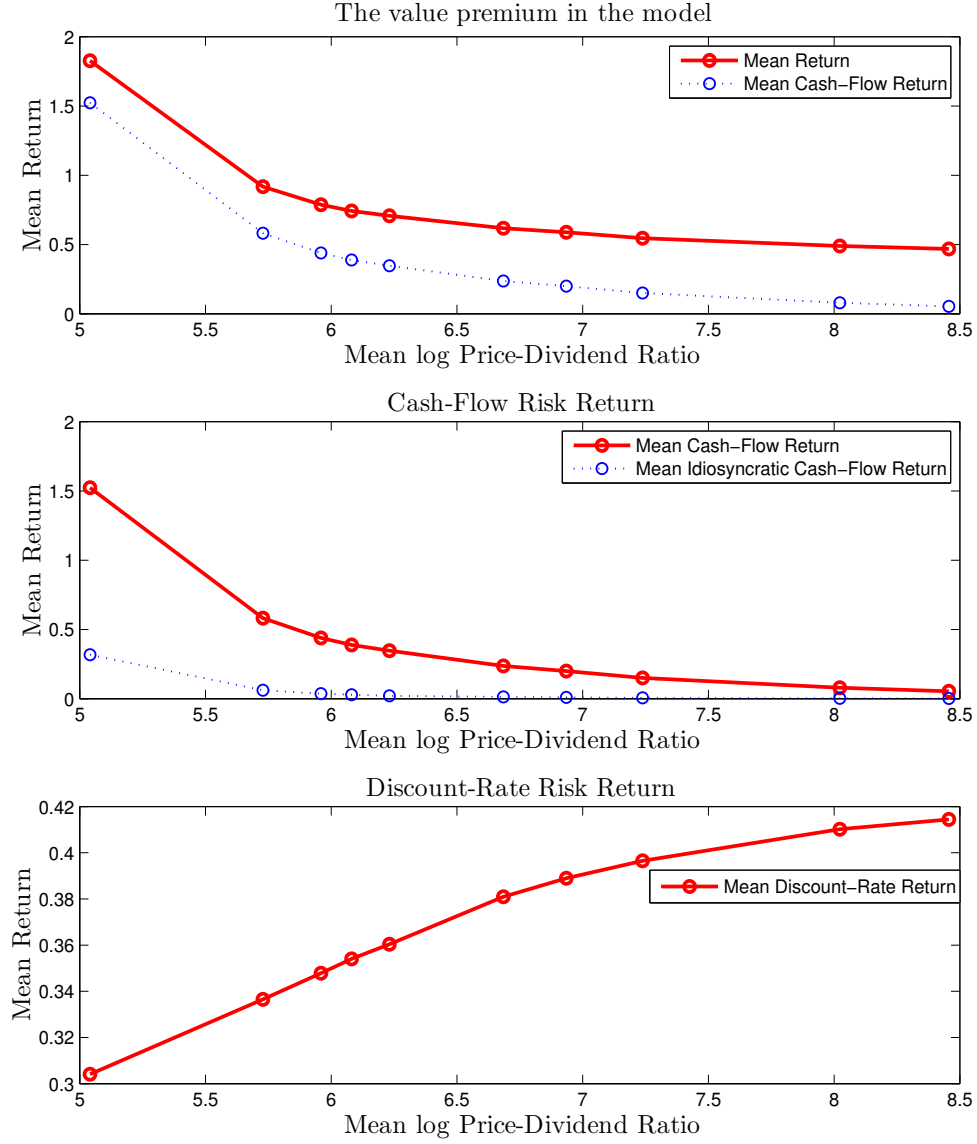


Figure 4.6: The Value Premium and Return Decomposition – The model is simulated with  $(\gamma, \delta) = (3.7, 1.9)$  and  $(h_1, h_2) = (0.0107, 0.089)$ . The top figure shows simulated cross-sectional average excess returns and cash flow risk returns. The middle figure shows the cash flow risk return and pure idiosyncratic cash flow risk return. And the bottom figure shows the discount-rate risk return. The simulation and return decompositions follow equilibrium equation in Proposition 2 and equations (3.8) and (3.9). Returns are expressed in percentage.

benchmark model, equilibrium excess return is not directly impacted by idiosyncratic cash flow risk since there is no belief difference.<sup>11</sup> Only the individual aggregate cash flow risk is emphasized regarding the cash flow risk. However as Table 4.4 shows, individual aggregate cash flow risk  $v_{s,A}$  do not differ much along the value decile. Thus the discount rate risk plays a major role in the cross-section, which leads to the counterfactual growth premium. The comparison between our model and the benchmark model highlights the importance of idiosyncratic cash flow risk in equilibrium cross-sectional pricing. Since the idiosyncratic cash flow risk is positively associated with belief difference in the cross-section, its effect is magnified by belief difference. Value firms have higher idiosyncratic cash flow risk and higher belief difference. Thus they are exposed to more (idiosyncratic) cash flow risk than growth firms. When the idiosyncratic cash flow risk is priced in equilibrium, the effect of cash flow risk dominates the effect of discount rate risk, hence the value premium arises.

#### 4.4.3 *Dissecting the Value Premium Anomaly*

In this section, we analyze the source of the value premium. Specifically we investigate the value premium using the implications from the discussion in 4.3. Economic implications in 4.3 indicate that most important features of cash flows are revealed through cross-sectional variability in the share ratios,  $\bar{s}/s_t$ , coefficients of variations in shares,  $CV(s_t)$ , and mean-reverting coefficients,  $\phi_s$ , since these three variables have a crucial information about how firms in value decile can be differentiated in terms of cash flow risks. As Table 4.6 and the top panel of Figure 4.7 show, our simulation is very similar to the one in Santos and Veronesi (2010). Thus the benchmark model is our version of Santos and Veronesi (2010).

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<sup>11</sup>Idiosyncratic cash flow risk actually exists in this case too. However it enters the model only through the share process, not through the pricing mechanism. Thus the effect of idiosyncratic cash flow risk is only indirect and its magnitude becomes almost negligible.



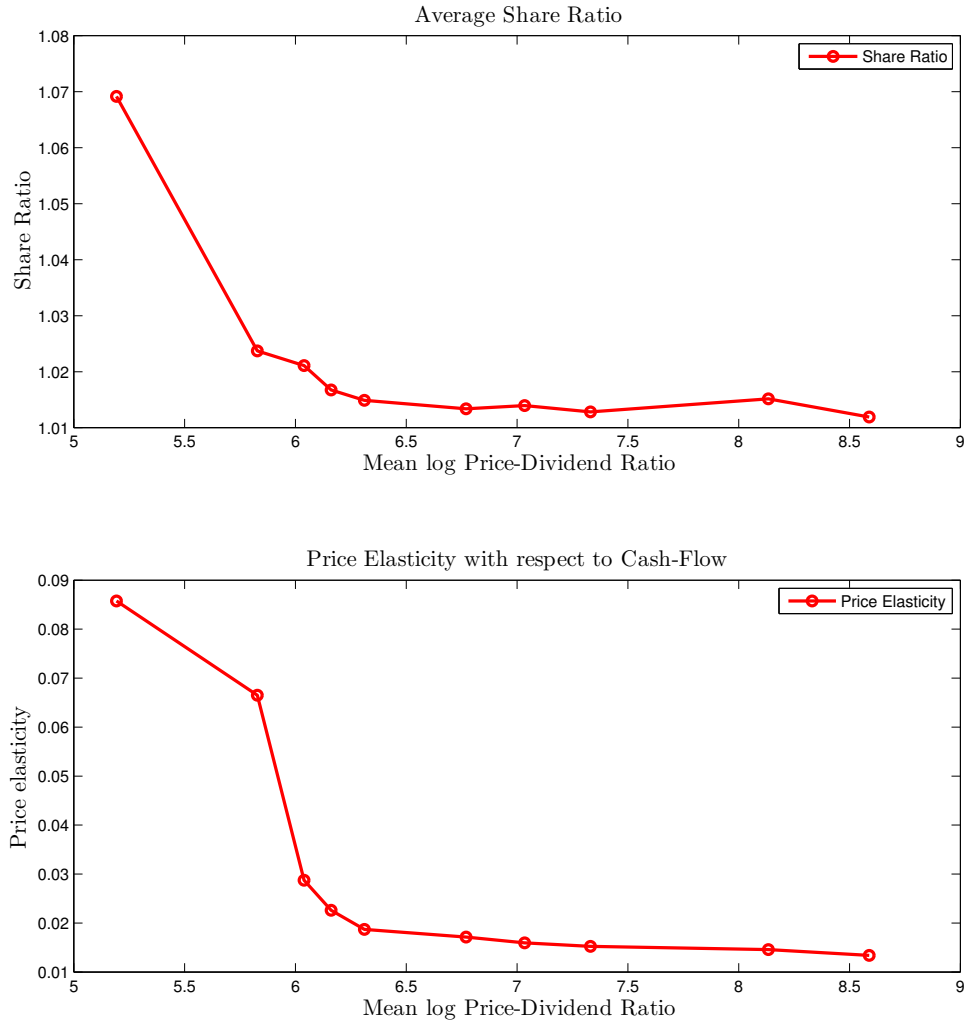


Figure 4.7: The Share Ratio and the Price Elasticity with respect to Cash Flow – The model is simulated with  $(\gamma, \delta) = (3.7, 1.9)$  and  $(h_1, h_2) = (0.0107, 0.089)$ . The top figure shows average share ratios along the value decile. Average share ratio is defined by the time-series mean of share ratios  $(\bar{s}/s_t)$  of firms in value decile. The bottom figure shows the inverse cash flow duration in value decile. The definition of the price elasticity with respect to cash flow is  $\frac{\partial P_s(t)/P_s(t)}{\partial D_s(t)/D_s(t)}$ .

model replicates empirical features of cash flow data.<sup>12</sup> Especially, our model generates data-consistent average share ratios and coefficients of variation of the share in value decile; simulated average share ratios and simulated coefficients of variation of shares are increasing from growth to value firms. Therefore we can apply the cash flow risk argument in 4.3 to our simulation results. As mentioned before, the share ratio is directly related to cash flow risk in the sense that higher share ratio of a firm leads to higher cash flow fluctuation as the share ratio is positively associated with the idiosyncratic cash flow risk parameter,  $v_{s,I}$ , higher coefficient of variation of the share, and a low mean-reverting coefficient of the share. Table 4.6 and Figure 4.7 show that there is positive (negative) relation between the share ratio and equilibrium expected excess return (equilibrium price-dividend ratio) in the cross-section. The fact that firms' payout policies affect the share ratio implies that the share ratio determines firms' exposures to the cash flow risk. Furthermore, most of cash flow risk comes from idiosyncratic component since individual aggregate cash flow risks,  $v_{s,A}$ 's, do not differ much in value decile. As was emphasized before, the effect of idiosyncratic cash flow risk is magnified through investors' belief difference, which generates the value premium in the cross-section. Therefore, our model simulation clearly shows how the value premium arises by revealing how firms are differently exposed to (idiosyncratic) cash flow risks in the cross-section through the lens of the share ratio, coefficient of variation of the share, and the mean-reverting coefficient of the share process.

The simulation result also confirms the conventional wisdom that growth(value) firms have higher (lower) cash flow duration as shown in Table 4.6 and Figure 4.7. According to simulated price elasticity with respect to cash flow, value firms are

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<sup>12</sup>We simulate firms with either  $\bar{s} = 0.05$  or with  $\bar{s}$  that is similar to estimated  $\bar{s}$  in the cross-section. In both cases, we could generate the same pattern of cash flow features.

much more sensitive to current cash flows than growth firms. The fact that the cash flow risk return and the discount rate risk return move oppositely, implies that the price elasticity with respect to cash flow and the price elasticity with respect to discount rate move oppositely although the latter cannot directly be computed. Therefore the price elasticity gives us an implicit information about the cash flow duration. Our simulation result supports the cash flow duration hypothesis such as Lettau and Wachter (2007), Lettau and Wachter (2011) or the timing of cash flows such as van Binsbergen, Brandt, and Koijen (2012).

Based on the discussion above, we can endogenously relate book-to-market variable to the value premium in a similar fashion that existing structural models such as Menzley, Santos, and Veronesi (2004), Santos and Veronesi (2010), Lettau and Wachter (2007), and Da (2009) describe. Aforementioned studies argue that stocks with high book-to-market ratios are associated with high cash flow risks. This positive association between the book-to-market ratio and the cash flow risk is also confirmed in our simulation since the dividend yield is positively associated with the share ratio as well as the coefficient of variation of the share. Therefore sorting stocks based on price-to-fundamentals (either book-to-market ratio or dividend yield) endogenously picks up stocks with higher cash flow risks in value decile so that the value premium arises since the variation of discount rate risk component is dominated by the variation of cash flow risk component. In addition to this, our model shows that the pricing of idiosyncratic cash flow risk can be crucial in generating the value premium. Idiosyncratic cash flow risk is positively associated with belief difference so that its impact is magnified from growth to value firms since value firms are exposed to higher investors' belief differences.<sup>13</sup>

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<sup>13</sup>Note that sorting stocks based on price-to-fundamentals is similar to sorting stocks based on the share ratio in value decile according to economic mechanism we described above. Since the share ratio is directly related to characteristics of firms' cash flows, sorting stocks based on average

We emphasize the importance of the pricing of idiosyncratic cash flow risk in comparison with Lettau and Wachter (2007) and Santos and Veronesi (2010) that are most close to our model. The primary goal of two studies is to generate the value premium while matching aggregate moments. Lettau and Wachter (2007) assume that shocks to the state variable in stochastic discount factor are uncorrelated with shocks to aggregate dividend. Thus shocks to stochastic discount factor can freely move to match key aggregate moments and suppress the effect of the discount rate risk in the cross-section. Value stocks covary more with front-loaded cash flows and growth stocks covary more with cash flows far in the future. As investors fear fluctuations of front-loaded cash flows, a value premium arises. Under a typical asset pricing model such as an external habit formation of Campbell and Cochrane (1999), the correlation between two aforementioned shocks is negative, which enables us to easily explain the aggregate equity premium and high return volatility with relatively low volatility in fundamentals. However, Lettau and Wachter (2007) show that given the negative correlation between two shocks, the growth premium arises as the effect of the discount rate risk is most pronounced. Indeed, Santos and Veronesi (2010) show this by studying a typical external habit formation model and investigate the cross-sectional pricing capability of cash flow risk that is defined by the covariance between the share process and aggregate cash flow process. They end up generating the growth premium in the cross-section since the discount rate risk is most pronounced. They counterfactually magnify the cross-sectional cash flow risk to generate the value premium. In summary, both Lettau and Wachter (2007) and Santos and Veronesi (2010) point out a common modeling challenge. In order to generate the value premium with prototypical asset pricing model in equilibrium, the effect of discount rate risk should be either suppressed (by magnifying the cash flow

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share ratios also well captures the cash flow risk in the data.

risk in the cross-section) or ignored (by muting the channel of discount rate effect in stochastic discount factor). Otherwise, the cash flow risk is not able to generate the value premium since the discount rate risk will still be most pronounced.

Our model complements both studies by shedding light on the modeling challenge mentioned above. The most crucial assumption in the model of Lettau and Wachter (2007) is that shocks to taste (or discount rate risk) are uncorrelated with shocks to the aggregate cash flow process. This differs from asset pricing models such as Campbell and Cochrane (1999), Menzley, Santos, and Veronesi (2004), and other models with external habit formation, since these models produce time-varying risk premium via current and past aggregate cash flows. In Lettau and Wachter (2007), shocks to the price of risk driven by a state variable unrelated to aggregate cash flows can match the aggregate risk premium without affecting the cross section. Note that the measure of investors' belief differences in our model appears in the stochastic discount factor and it has zero correlation with the aggregate cash flow process. Instead, being associated with idiosyncratic cash flow risk, it reduces the effect of discount rate risk in the cross-section so that the value premium arises. In doing so, the idiosyncratic cash flow risk *positively* affects the equilibrium expected excess return. Therefore our model provides a modeling justification on the use of taste shocks in the stochastic discount factor in the reduced form model of Lettau and Wachter (2007). Besides, our model also explains the puzzle of magnified cash flow risk in Santos and Veronesi (2010) in generating the value premium. In their result, individual aggregate cash flow risk does not differ much in value decile so that the growth premium arises as the discount rate risk dominates. Our model offers the equilibrium pricing of idiosyncratic cash flow risk that is increasing from growth to value stocks. It turns out that the idiosyncratic cash flow risk accounts for the most of cross-sectional return variation.

## 4.5 Robustness

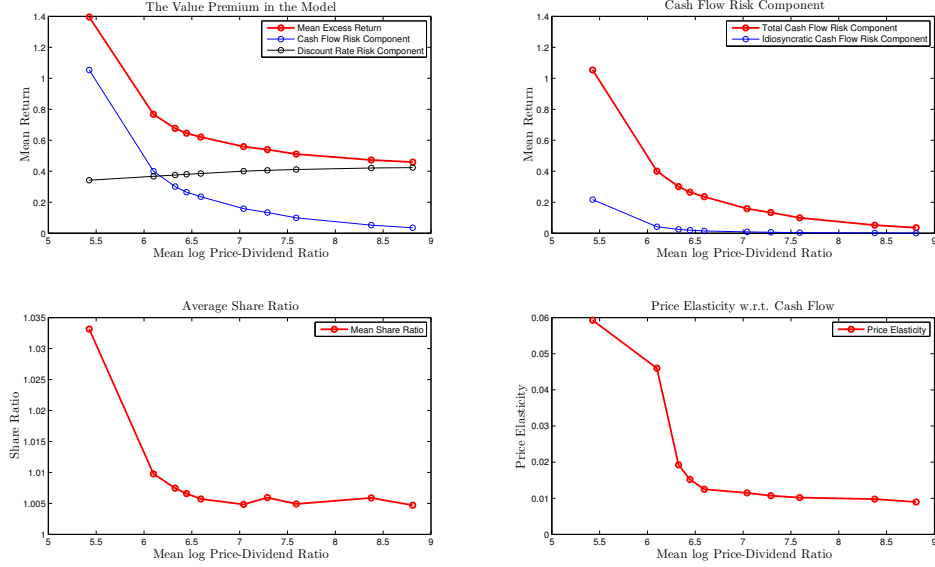


Figure 4.8: Robustness 1 – The model is simulated with  $(\gamma, \delta) = (1.9, 0.6)$  and  $(h_1, h_2) = (0.0107, 0.089)$ . Upper-left shows simulated cross-sectional average excess returns, cash flow risk returns, and discount-rate risk returns. Upper-right shows idiosyncratic cash flow risk return and total cash flow risk return. Mid-left shows the relation between average share ratios and price-dividend ratios that are driven by share ratios. Mid-right shows the relation between average share ratios and returns that are driven by share ratios. Lower-left shows average share ratios of decile portfolios. Lower-right shows the price elasticity with respect to the cash flow alongside decile portfolios.

Figure 4.8 and 4.9 show the robustness of the simulation result of our model. Note that  $\gamma$  and  $\delta$  are two main parameters governing investors' preferences. In calibrations,  $h_1$  and  $h_2$  are almost the same across wide range of combinations of  $(\gamma, \delta)$ . Thus we vary values of  $\gamma$  and  $\delta$  in the simulation to check the robustness of the simulation results. Figure 4.8 and figure 4.9 show almost the same result compared to

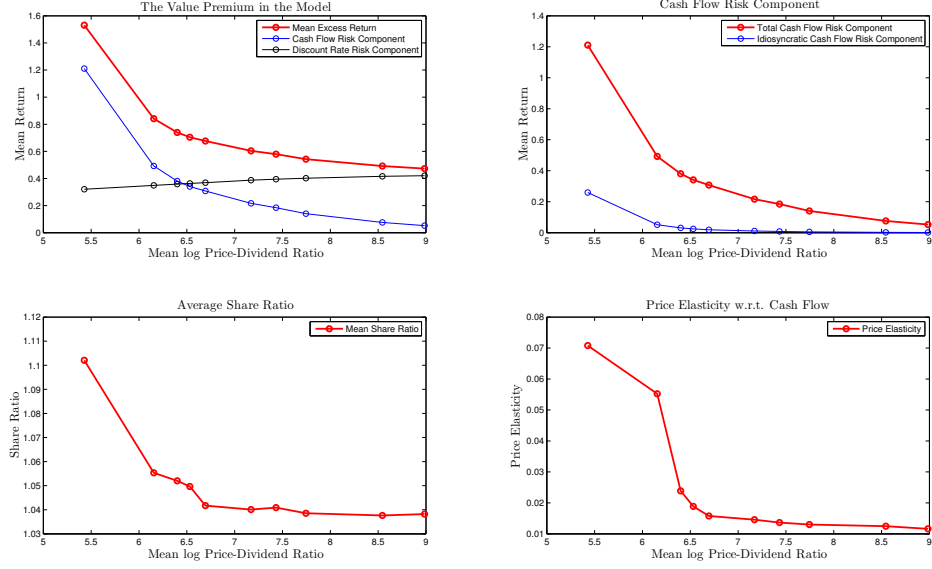


Figure 4.9: Robustness 2 – The model is simulated with  $(\gamma, \delta) = (5.3, 1.9)$  and  $(h_1, h_2) = (0.0107, 0.089)$ . Upper-left shows simulated cross-sectional average excess returns, cash flow risk returns, and discount-rate risk returns. Upper-right shows idiosyncratic cash flow risk return and total cash flow risk return. Mid-left shows the relation between average share ratios and price-dividend ratios that are driven by share ratios. Mid-right shows the relation between average share ratios and returns that are driven by share ratios. Lower-left shows average share ratios of decile portfolios. Lower-right shows the price elasticity with respect to the cash flow alongside decile portfolios.

original model simulation, which confirms the robustness of our model.<sup>14</sup> In addition to varying key aggregate parameters, we also carry a thought experiment in which the values of idiosyncratic cash flow risk parameters are fixed across all firms, but with the same parameter values of belief differences as before. This set up gives us a similar result with slightly reduced cross-sectional return variations.<sup>15</sup> This confirms that difference in beliefs is the channel through which idiosyncratic cash flow risks work in the cross-section.

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<sup>14</sup>We carried out total of 8 combinations of  $(\gamma, \delta)$ . Figures 4.8 and 4.9 show the most extreme cases. Other combinations of  $(\gamma, \delta)$  show the same results that can be obtained upon request.

<sup>15</sup>If we magnify the values of belief differences, we still get almost identical result to the original simulation.



## 5. CONCLUSION

In this paper, we show that differences in investors' beliefs on firms' cash flows play an instrumental role in explaining the return portfolios sorted by the book-to-market ratio. In the data, we find that aggregate cash flow risks do not differ much in value deciles, yet the fluctuation of cash flow shares increases from the growth stocks to the value stocks. This implies that idiosyncratic cash flow risk should be high for the value stocks. Our model with belief heterogeneity allows the idiosyncratic cash flow risk to be priced in equilibrium. Furthermore our model states that idiosyncratic cash flow risk and belief difference are positively associated so that the effect of idiosyncratic cash flow risk can be magnified along the value deciles, if the value stocks are more prone to the divergence of opinions. That is, the value stocks can have higher expected returns than the growth stocks due to the (idiosyncratic) cash flow risk along with higher belief difference. Our empirical result shows that the value stocks indeed have higher degrees of belief difference, and we can quantitatively produce the size of value premium observed by the data.

Related, when the channel of belief difference is turned off, a growth premium appears in the model. Given that the growth stocks are those that pay more in the future, investors can request a premium for bearing higher discount risk. This suggests that without an additional device, conventional measure of cash flow risks may be insufficient to dominate discount risk to explain a value premium, and our model provides one route to resolve this issue. It would be interesting to study how heterogeneous beliefs are associated with other asset pricing anomalies, which we leave as future tasks.

## REFERENCES

- Abel, Andrew B., 1989, Asset prices under heterogeneous beliefs: Toward a resolution of the equity premium puzzle, *Rodney L. White Center for Financial Research, Working Paper*.
- , 1999, Risk premia and term premia in general equilibrium, *Journal of Monetary Economics* 43, 3–33.
- Acemoglu, Daron, Victor Chernozhukov, and Muhamet Yildiz, 2007, Learning and disagreement in an uncertain world, *Working paper*.
- Aiyagari, Rao S., 1994, Uninsured idiosyncratic risk and aggregate saving, *Quarterly Journal of Economics* 109, 659–684.
- Anderson, Evan, Eric Ghysels, and Jennifer Juergens, 2005, Do heterogeneous beliefs matter for asset pricing?, *Review of Financial Studies* 18, 875–924.
- Avramov, Doron, Scott Cederburg, and Satadru Hore, 2012, Cross-sectional asset pricing puzzles: A long-run perspective, *Working paper*.
- Babenko, Ilona, Oliver Boguth, and Yuri Tserlukevich, 2013, Idiosyncratic cash flows and systematic risk, *Working paper*.
- Bansal, Ravi, Robert Dittmar, and Christian Lundblad, 2005, Consumption, dividends, and the cross-section of equity returns, *Journal of Finance* 60, 1639–1672.
- Bansal, Ravi, and Amir Yaron, 2004, Risks for the long run: A potential resolution of asset pricing puzzles, *Journal of Finance* 59, 1481–1509.

- Basak, Suleyman, 2000, A model of dynamic equilibrium asset pricing with extraneous risk, *Journal of Economic Dynamics and Control* 24, 63–95.
- , 2005, Asset pricing with heterogeneous beliefs, *Journal of Banking and Finance* 29, 2849–2881.
- , and Domenico Cuoco, 1998, An equilibrium model with restricted stock market participation, *Review of Financial Studies* 11, 309–341.
- Basak, Suleyman, and Michael Gallmeyer, 2003, Capital market equilibrium with differential taxation, *Review of Financial Studies* 7, 121–159.
- Bhamra, Harjoat S., and Raman Uppal, 2010, Asset prices with heterogeneity in preferences and beliefs, *Working paper*.
- Buraschi, Alexander, and Alexander Jiltsov, 2006, Model uncertainty and option markets with heterogeneous agents, *Journal of Finance* 61, 2841–2897.
- Campbell, John, Christopher Polk, and Tuomo Vuolteenaho, 2010, Growth or glamour? fundamentals and systematic risk in stock returns, *Review of Financial Studies* 23, 305–344.
- Campbell, John Y., and John H. Cochrane, 1999, By force of habit: A consumption-based explanation of aggregate stock market behavior, *Journal of Political Economy* 107, 205–251.
- Campbell, John Y., and Tuomo Vuolteenaho, 2004, Bad beta, good beta, *The American Economic Review* 94, 1249–1275.
- Carlin, Bruce I., Francis A. Longstaff, and Kyle Matoba, 2013, Disagreement and asset prices, *Working paper*.

- Chen, Hui, and Scott Joslin, 2012, Generalized transform analysis of affine processes and applications in finance, *Review of Financial Studies* 25, 2225–2256.
- Chen, Huafeng (Jason), 2013, Do cash flows of growth stocks really grow faster?, *Working paper*.
- Chen, Joseph, Harrison Hong, and Jeremy Stein, 2002, Breadth of ownership and stock returns, *Journal of Financial Economics* 66, 171–205.
- Choi, Seung Mo, Shane Johnson, Hwagyun Kim, and Changwoo Nam, 2013, Dividend policy, investment, and stock returns, *TAMU Mays Business School Working paper*.
- Cohen, Randolph, Christopher Polk, and Tuomo Vuolteenaho, 2009, The price is (almost) right, *Journal of Finance* 64, 2739–2782.
- Constantinides, Goerge M., 1990, Habit formation: A resolution of the equity premium puzzle, *Journal of Political Economy* 98, 519–543.
- Cuoco, Domenico, and Hua He, 1994, Dynamic equilibrium in infinite-dimensional economies with incomplete financial markets, *Working paper*.
- Da, Zhi, 2009, Cash flow, consumption risk, and the cross-section of stock returns, *Journal of Finance* 64, 923–956.
- , and Mitchell Craig Warachka, 2009, Cashflow risk, systematic earnings revisions, and the cross-section of stock returns, *Journal of Financial Economics* 94, 448–468.
- David, Alexander, 2008, Heterogeneous beliefs, speculation, and the equity premium, *Journal of Finance* 63, 41 – 83.

- Dechow, Patricia M., Richard G. Sloan, and Mark T. Soliman, 2004, Implied equity durations: A new measure of equity risk, *Review of Accounting Studies* 9, 197–228.
- Detemple, Jerome, and Shashidhar Murthy, 1994, Intertemporal asset pricing with heterogeneous beliefs, *Journal of Economic Theory* 62, 294–320.
- Detemple, Jerome, and Angel Serrat, 2003, Dynamic equilibrium with liquidity constraints, *Review of Financial Studies* 16, 597–629.
- Detemple, Jerome, and Fernando Zapatero, 1991, Asset prices in an exchange economy with habit formation, *Econometrica* 59, 1633–1657.
- Diether, Karl B., Christopher J. Malloy, and Anna Scherbina, 2005, Differences of opinion and the cross section of stock returns, *Journal of Finance* 57, 2113–2141.
- Duffie, Darrell, Jun Pan, and Kenneth Singleton, 2000, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* 68, 1343–1376.
- Dumas, Bernard, Alexander Kurshev, and Raman Uppal, 2009, Equilibrium portfolio strategies in the presence of sentiment risk and excess volatility, *Journal of Finance* 64, 579–629.
- Dybvig, Philip H., and Chifu Huang, 1988, Nonnegative wealth, absence of arbitrage, and feasible consumption plans, *Review of Financial Studies* 1, 377–401.
- Ehling, Paul, Michael F. Gallmeyer, Christian Heyerdahl-Larsen, and Philipp K. Illeditsch, 2012, Disagreement about inflation and the yield curve, *Working paper*.
- Franke, Gunter, Richard G. Stapleton, and Marti G. Subrahmanyam, 1993, Idiosyncratic risk, sharing rules and the theory of risk bearing, *INSEAD Working Paper*.

- Gabaix, Xavier, 2009, Linearity-generating processes: A modelling tool yielding closed forms for asset prices, *Working paper*.
- Gallmeyer, Michael, and Burton Hollifield, 2008, An examination of heterogeneous beliefs with a short-sale constraint in a dynamic economy, *Review of Finance* 12, 323–364.
- Goetzmann, William, and Massimo Massa, 2005, Dispersion of opinion and stock returns, *Journal of Financial Markets* 8, 324–349.
- Grullon, Gustavo, and Roni Michaely, 2002, Dividends, share repurchases, and the substitution hypothesis, *Journal of Finance* 57, 1649–1684.
- Hansen, Lars Peter, John Heaton, and Nan Li, 2008, Consumption strikes back? measuring long-run risk, *Journal of Political Economy* 116, 260–302.
- Harris, Milton, and Arthur Raviv, 1993, Differences of opinion make a horse race, *Review of Financial Studies* 6, 473 – 506.
- Harrison, Michael J., and David M. Kreps, 1978, Speculative investor behavior in a stock market with heterogeneous expectations, *Quarterly Journal of Economics* 93, 323 – 336.
- Heaton, John, and Deborah J. Lucas, 1996, Evaluating the effects of incomplete markets on risk sharing and asset pricing, *Journal of Political Economy* 104, 443–487.
- Johnson, Timothy, 2004, Forecast dispersion and the cross-section of expected returns, *Journal of Finance* 59, 1957–1978.
- Jouini, Elyés, and Clotilde Napp, 2007, Consensus consumer and intertemporal asset pricing with heterogeneous beliefs, *Review of Economic Studies* 74, 1149–1174.

- Kahn, James A., 1990, Moral hazard, imperfect risk-sharing, and the behavior of asset returns, *Journal of Monetary Economics* 26, 27–44.
- Kiku, Dana, 2007, Is the value premium a puzzle?, *Working paper*, Wharton.
- Lettau, Martin, and Jessica Wachter, 2007, Why is long-horizon equity less risky? a duration-based explanation of the value premium, *Journal of Finance* 62, 55 – 92.
- , 2011, The term structures of equity and interest rates, *Journal of Financial Economics* 101, 90–113.
- Lintner, John, 1965, The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets, *Review of Economics and Statistics* 47, 13–37.
- Liptser, Robert S., and Albert N. Shiryaev, 2001, *Statistics of Random Processes I* (Springer; 2nd rev. and exp. ed. 2001 edition: New York).
- Lucas, Deborah J., 1994, Asset pricing with undiversifiable income risk and short sales constraints: Deepening the equity premium puzzle, *Journal of Monetary Economics* 34, 325–341.
- Massari, Filippo, 2012, Price dynamics and market selection, *Working paper*.
- Menzley, Lior, Tano Santos, and Pietro Veronesi, 2004, Understanding return predictability, *Journal of Political Economy* 112, 1–47.
- Miller, Edward, 1977, Risk, uncertainty, and divergence of opinion, *Journal of Finance* 32, 1151–1168.
- Morris, Stephen, 1994, Trade with heterogeneous prior beliefs and asymmetric information, *Econometrica* 60, 1327–1347.

- , 1996, Speculative investor behavior and learning, *Quarterly Journal of Economics* 111, 1111–1133.
- Park, C., 2005, Stock return predictability and the dispersion in earnings forecasts, *Journal of Business* 78, 2351–2376.
- Qu, Shisheng, Laura Starks, and Hong Yan, 2004, Risk, dispersion of analyst forecasts and stock returns, *Working paper*.
- Santos, Tano, and Pietro Veronesi, 2006, Labor income and predictable stock returns, *Review of Financial Studies* 19, 1–44.
- , 2010, Habit formation, the cross section of stock returns and the cash-flow risk puzzle, *Journal of Financial Economics* 98, 385 – 413.
- Sims, Christopher A., 2003, Implications of rational inattention, *Journal of Monetary Economics* 50, 665–690.
- , 2006, Rational inattention: Beyond the linear-quadratic case, *American Economic Review Papers and Proceedings* 96, 158–163.
- Stephens, Clifford P., and Michael S. Weisbach, 1998, Actual share reacquisitions in open market repurchase programs, *Journal of Finance* 53, 313–334.
- Storesletten, Kjetil, Chris Telmer, and Amir Yaron, 2007, Asset pricing with idiosyncratic risk and overlapping generations, *Review of Economic Dynamics* 10, 519 – 548.
- Telmer, Chris I., 1993, Asset pricing puzzles and incomplete market, *Journal of Finance* 48, 1803–1832.



- van Binsbergen, Jules, Michael Brandt, and Ralph Koijen, 2012, On the timing and pricing of dividends, *American Economic Review* 102, 1596–1618.
- Varian, Hal R., 1985, Divergence of opinion in complete markets: A note, *Journal of Finance* 40, 309–317.
- , 1989, Differences of opinion in financial markets, *Chapter 1 in Financial Risk: Thoery, Evidence, and Implications*, edited by Courtney C. Stone, Kluwer Academic Publishers, Boston, MA.
- Weinbaum, David, 2009, Investor heterogeneity, asset pricing and volatility dynamics, *Journal of Economic Dynamics and Control* 33, 1379–1397.
- Xiong, Wei, and Lin Peng, 2006, Investor inattention, overconfidence and category learning, *Journal of Financial Economics* 80, 563–602.
- Yang, Wei, 2007, Time-varying exposure to long-run consumption risk, *Working paper*, University of Rochester.
- Yu, Jialin, 2011, Disagreement and return predictability of stock portfolios, *Journal of Financial Economics* 99, 162 – 183.
- Zapatero, Fernando, 1998, Effects of financial innovations on market volatility when beliefs are heterogeneous, *Journal of Economic Dynamics and Control* 22, 597–626.
- Zhang, Lu, 2005, The value premium, *Journal of Finance* 60, 67–104.
- Zhang, X. Frank, 2006, Information uncertainty and stock returns, *Journal of Finance* 61, 105–136.

## APPENDIX A

### BASICS OF SHARE PROCESSES

In this appendix we investigate details of the share process and the individual cash flow processes that are driven from the former. For our study, we just want a single firm embedded in the economy so that we simplify the share process down to one small firm. The specification of the share process for this purpose is the same as the one given in the paper. We start from the basic cash flow process of the asset corresponding to the share process,  $s_t$ . There always is another asset corresponding to the share process,  $(1 - s_t)$ . Following MSV (2004), we take these two *primitive* cash flow processes as following;

$$\begin{aligned}\frac{dD_s}{D_s} &= \mu_s dt + v_{s,A} dB_A + v_{s,I} dB_I, \\ \frac{dD_{(1-s)}}{D_{(1-s)}} &= \mu_{(1-s)} dt + v_{(1-s),A} dB_A + v_{(1-s),I} dB_I\end{aligned}\tag{A.1}$$

, where  $v_{ij}$  for  $i = s, (1 - s)$  and  $j = A, I$  are taken as constants for simplicity. This is the same setting that was adopted by Menzley, Santos, and Veronesi (2004) when there are only two assets in the economy. Note that the cash flow's dependence on the idiosyncratic risk is measured by coefficients  $v_{s,I}$  and  $v_{(1-s),I}$  on the same idiosyncratic Brownian risk.<sup>1</sup> The aggregate risk and the idiosyncratic risk are assumed to be independent so that  $B_A$  and  $B_I$  are given as independent Brownian motions.

Since the share,  $s_t$ , is defined as  $D_s/D \equiv D_s/(D_s + D_{(1-s)})$ , by applying Ito lemma to the latter part, we get the diffusion process of the share process  $s_t$ . Diffusion

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<sup>1</sup>Similar method was adopted by Hugonnier and Verrada (2011).

coefficients are given by

$$\begin{aligned} s_t \left( v_{s,A} - s_t v_{s,A} - (1 - s_t) v_{(1-s),A} \right), \\ s_t \left( v_{s,I} - s_t v_{s,I} - (1 - s_t) v_{(1-s),I} \right), \end{aligned} \quad (\text{A.2})$$

for  $dB_A$  and  $dB_I$  respectively. By taking the first coefficient as  $s_t \sigma_{s,A}$  and the second one as  $s_t \sigma_{s,I}$ , and by imposing the mean-reverting structure on the drift term, we get the share process given in the paper.

Following Menzley, Santos, and Veronesi (2004), restrictions are imposed to guarantee that dividends are positive and the share is positive. In the case of two risky assets, restrictions are given by

$$\begin{aligned} \bar{s} < 1 \quad \text{and} \quad \overline{(1-s)} < 1, \\ \phi_s > 0, \quad \overline{(1-s)} \cdot \phi_{(1-s)} \quad \text{and} \quad \phi_{(1-s)} > 0, \quad \bar{s} \cdot \phi_s. \end{aligned} \quad (\text{A.3})$$

Since  $\sigma_{s,j}$  and  $\sigma_{(1-s),j}$  for  $j = A, I$  are invariant to adding the same vector to each  $v_s$  or  $v_{(1-s)}$ , we can normalize  $v_s$  and  $v_{(1-s)}$  so that, as in Menzley, Santos, and Veronesi (2004), we have the following equation (identification condition):

$$\bar{s} \mathbf{v}_s + (1 - \bar{s}) \mathbf{v}_{(1-s)} = 0, \quad (\text{A.4})$$

where  $\mathbf{v}_i$  is the row vector of  $v_{i,A}$  and  $v_{i,I}$  for  $i = s, (1-s)$ .

We now derive the individual cash flow process. Given the share process, an individual cash flow  $D_s(t)$  is defined as  $D_s(t) \equiv s_t D_t$ . By applying Ito lemma to  $s_t D_t$ , we can derive the diffusion process of an individual cash flow  $D_s(t)$ .

$$\frac{dD_s(t)}{D_s(t)} = \mu_{D_s}(t) dt + \sigma_{D_s,A}(t) dB_A(t) + \sigma_{D_s,I}(t) dB_I(t), \quad (\text{A.5})$$

where

$$\begin{aligned}
\mu_{D_s}(t) &\equiv \mu_D + \phi_s \left( \frac{\bar{s}}{s_t} - 1 \right) + \theta_s^{CF} - s_t \theta_s^{CF} - (1 - s_t) \theta_{(1-s)}^{CF}, \\
\sigma_{D_s,A}(t) &\equiv \sigma_{D,A} + \sigma_{s,A}(t), \\
\sigma_{D_s,I}(t) &\equiv \sigma_{s,I}(t),
\end{aligned} \tag{A.6}$$

where  $\theta_s^{CF} \equiv \sigma_{D,A} v_{s,A}$ , and  $\theta_{(1-s)}^{CF} \equiv \sigma_{D,A} v_{s,A}$ . The covariance between share and the aggregate dividend(consumption) growth is given by

$$Cov_t \left( \frac{ds_t}{s_t}, \frac{dD_t}{D_t} \right) = \theta_s^{CF} - [\theta_s^{CF} s_t + \theta_{(1-s)}^{CF} (1 - s_t)]. \tag{A.7}$$

By computing the unconditional covariance from the data, we obtain

$$E \left[ Cov_t \left( \frac{ds_t}{s_t}, \frac{dD_t}{D_t} \right) \right] = v_{s,A} \cdot \sigma_{D,A} \equiv \theta_s^{CF}, \tag{A.8}$$

due to the identification condition of the share process (A.4). Thus we can pin down  $v_{s,A}$  by computing unconditional covariance between the share process and aggregate cash flow process in the data. Note that the conditional variance of individual cash flow process is given by

$$var_t \left( \frac{dD_s(t)}{D_s(t)} \right) = [\sigma_{D,A} + \sigma_{s,A}(t)]^2 + [\sigma_{s,I}(t)]^2. \tag{A.9}$$

When the conditional variance is evaluated at  $s_t = \bar{s}$ , we get

$$(\sigma_{D,A} + v_{s,A})^2 + (v_{s,I})^2. \tag{A.10}$$

Since  $v_{s,A}$  is already pinned down from unconditional covariance between the share

process and the aggregate cash flow process, one can recover  $v_{s,I}$  based on a individual total cash flow volatility. Finally by using identifiability condition (A.4), we can derive

$$v_{(1-s),j} = -\frac{\bar{s}v_{s,j}}{1-\bar{s}}, \quad j = A, I, \quad (\text{A.11})$$

for both aggregate and idiosyncratic terms.

## APPENDIX B

### CALIBRATION

In this appendix, we explain the calibration of our model. Calibration is applied to the aggregate moments, i.e., mean aggregate market excess return, aggregate market volatility, mean aggregate price-to-dividend ratio, Sharpe ratio, mean riskless rate, and the volatility of riskless rate. Note that the aggregate price-dividend ratio is represented as

$$\frac{P_t}{D_t} = \frac{1}{h_1} - \frac{\bar{H}}{H_t}. \quad (\text{B.1})$$

And the diffusion process of aggregate excess market return is represented as

$$dR_t = \mu_R dt + \sigma_{RA} dB_A, \quad (\text{B.2})$$

where

$$\begin{aligned} \mu_R &\equiv (\sigma_{DA} - h_2)^2 + \frac{h_2}{h_1} (\sigma_{DA} - h_2) \frac{D_t}{P_t}, \\ \sigma_{RA} &\equiv (\sigma_{DA} - h_2) + 2 \frac{h_2}{h_1} \frac{D_t}{P_t}. \end{aligned} \quad (\text{B.3})$$

Note that the stochastic discount factor is also represented by  $H_t$ . Thus when we match the unconditional theoretical aggregate moments to their sample counterpart, we need the probability density function of  $H_t$ . The stationary density function of  $H_t$  is given by

$$f(H) = \frac{\exp \left[ -2b \left( \frac{\bar{H}}{H} \right) \right] \times H^{-2b-2}}{\int_0^\infty \exp \left[ -2b \left( \frac{\bar{H}}{h} \right) \right] \times h^{-2b-2}}, \quad (\text{B.4})$$

where  $b \equiv h_1/h_2^2$ . By using the stationary density  $f(H)$ , we can compute  $E[dR_t]$ ,  $E[r_f]$ ,  $E\left[\frac{P_t}{D_t}\right]$ ,  $E\left[\sigma_{r_f}^2\right]$ , and  $\frac{E[dR_t]}{E[dR_t^2]}$ . We proceed by matching mean aggregate price-dividend ratio first and then matching other aggregate moments to their sample counterpart, i.e., the average of aggregate excess return, the average riskless rate, the volatility of riskless rate, and the aggregate Sharpe ratio. In matching first and second moments of aggregate excess return, we focus on matching Sharpe ratio.

## APPENDIX C

### PROOFS

**Derivation of  $\lambda_t$  process.** By applying Ito's lemma to the definition of  $\lambda_t$  process, we get

$$\frac{d\lambda(t)}{\lambda(t)} = \left[ -r_t + \mu_{\xi^{(1)-1}(t)} - \theta_A^2(t) - \theta_I^{(1)}(t)\theta_I^{(2)}(t) \right] dt + \theta_I^{(1)}(t)dB_I^{(1)}(t) - \theta_I^{(2)}(t)dB_I^{(2)}(t), \quad (\text{C.1})$$

where

$$\mu_{\xi^{(1)-1}}(t) \equiv r_t + \theta_A^2(t) + \theta_I^{(1)2}(t). \quad (\text{C.2})$$

Diffusion terms are expressed as  $\bar{\eta}_t dB_I^{(2)}(t) - \bar{\eta}_t \theta_I^{(1)}(t) dt$ . And the drift term in (C.1) and  $-\bar{\eta}_t \theta_I^{(1)}(t) dt$  are summed up to zero since  $\bar{\eta}_t = \eta_t^{(2)} - \eta_t^{(1)} = \theta_I^{(1)}(t) - \theta_I^{(2)}(t)$ . Therefore

$$\frac{d\lambda_t}{\lambda_t} = \bar{\eta}_t dB_I^{(2)}(t). \quad (\text{C.3})$$

□

**Derivation of market prices of risks.** Since the financial market is dynamically complete, we can use price processes of the market portfolio and the asset



with the share process  $s_t$  for determining the market prices of risks. Thus

$$\begin{aligned}
\begin{pmatrix} \theta_A \\ \theta_I^{(k)} \end{pmatrix} &= \begin{pmatrix} \sigma_{P,A} & 0 \\ \sigma_{P_s,A} & \sigma_{P_s,I} \end{pmatrix}^{-1} \begin{pmatrix} \mu_P - r \\ \mu_{P_s}^{(k)} - r \end{pmatrix} \begin{pmatrix} dB_A \\ dB_I^{(k)} \end{pmatrix} \\
&= \frac{1}{\sigma_{P,A}\sigma_{P_s,I}} \begin{pmatrix} \sigma_{P_s,I}(\mu_P - r) \\ -\sigma_{P_s,A}(\mu_P - r) + \sigma_{P,A}(\mu_{P_s}^{(k)} - r) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\mu_P - r}{\sigma_{P,A}} \\ -\frac{\sigma_{P_s,A}}{\sigma_{P_s,I}} \frac{1}{\sigma_{P,A}} (\mu_P - r) + \frac{1}{\sigma_{P_s,I}} (\mu_{P_s}^{(k)} - r) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\mu_P - r}{\sigma_{P,A}} \\ -\frac{\sigma_{P_s,A}}{\sigma_{P_s,I}} \theta_A + \frac{1}{\sigma_{P_s,I}} (\mu_{P_s} - r) - \eta^{(k)} \end{pmatrix}
\end{aligned} \tag{C.4}$$

□

**Proof of Proposition 1.** Note that the equilibrium stock price can be represented by using either one of state price densities across investors due to the existence of  $\lambda_t$  process.<sup>1</sup> The equilibrium price of an asset with the share process  $s_t$  is represented

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<sup>1</sup>The existence of  $\lambda_t$  process implies that when we compute the equilibrium price using investor 1's state price density, this equilibrium price will be the same as the equilibrium price that we compute by using investor 2's state price density. The reason is that  $\lambda$ -process is not only the measure of degree of belief difference, but also the Radon-Nikodym derivative that relates two investors' state price densities so that the price that is computed with one state price density will end up with the same as the price that is computed with the other state price density.

as

$$\begin{aligned}
P_s(t) &= E_t \left[ \int_t^\infty \frac{\xi_\tau^{(2)}}{\xi_t^{(2)}} s_\tau D_t d\tau \right] \\
&= \frac{1}{\left(1 + \lambda_t^{1/\gamma}\right)^\gamma \left(\frac{1}{X_t}\right)^{1-\gamma} D_t^{-\gamma}} E_t \left[ \int_t^\infty \left(1 + \lambda_\tau^{1/\gamma}\right)^\gamma \left(\frac{1}{X_\tau}\right)^{1-\gamma} D_\tau^{-\gamma} s_\tau D_\tau d\tau \right] \\
&= \frac{s_t D_t}{s_t \left(1 + \lambda_t^{1/\gamma}\right)^\gamma \left(\frac{D_t}{X_t}\right)^{1-\gamma}} E_t \left[ \int_t^\infty s_\tau \left(1 + \lambda_\tau^{1/\gamma}\right)^\gamma \left(\frac{D_\tau}{X_\tau}\right)^{1-\gamma} d\tau \right] \\
&= \frac{s_t D_t}{q_t} E_t \left[ \int_t^\infty q_\tau d\tau \right] \\
&= \frac{D_s(t)}{q_t} E_t \left[ \int_t^\infty q_\tau d\tau \right]
\end{aligned} \tag{C.5}$$

, where

$$\begin{aligned}
q_t &\equiv s_t z_t H_t, \\
z_t &\equiv \left(1 + \lambda_t^{1/\gamma}\right)^\gamma, \\
H_t &\equiv \left(\frac{D_t}{X_t}\right)^{1-\gamma}.
\end{aligned} \tag{C.6}$$

Now we define the diffusion process  $H_t$ . We name  $\bar{H}/H_t$  as a ‘‘Habit Ratio’’. In Campbell and Cochrane (1999) and Santos and Veronesi (2010), the consumption surplus ratio  $S_t^\gamma$  is the proxy for the shock to aggregate discount rate as an element in stochastic discount factor. In our model, however, a similar variable  $H_t$  does not directly proxy aggregate discount rate since it does not induce the time-varying risk preference. It represents the aggregate shock to stochastic discount factor as well as an indicator of economic conditions. When this ratio is high, the economy is in a good condition and vice versa. By applying Ito’s lemma to the process  $H_t \equiv (D_t/X_t)^{1-\gamma}$ ,

we get

$$d\left(\frac{D_t}{X_t}\right)^{1-\gamma} = (1-\gamma)\left(\frac{D_t}{X_t}\right)^{1-\gamma} \left\{ \left[ \mu_D - \lambda \left( \frac{D_t}{X_t} - 1 \right) - \frac{1}{2} \gamma \sigma_{DA}^2 \right] dt + \sigma_{DA} dB_A \right\}. \quad (\text{C.7})$$

Following Campbell and Cochrane (1999), Menzley, Santos, and Veronesi (2004), and Santos and Veronesi (2010), we assume a simpler process of  $H_t$  as follows<sup>2</sup>.

$$dH_t = h_1(\bar{H} - H_t)dt + h_2 H_t dB_A(t). \quad (\text{C.8})$$

Note that the diffusion process of  $\lambda_t^{1/\gamma}$  is given by

$$\frac{d\lambda_t^{1/\gamma}}{\lambda_t^{1/\gamma}} = \alpha_1(t)dt + \alpha_2(t)dB_I^{(2)}, \quad (\text{C.9})$$

where

$$\alpha_1(t) \equiv \frac{1}{2} \frac{1}{\gamma} \left( \frac{1}{\gamma} - 1 \right) \bar{\eta}_t^2, \quad \alpha_2(t) \equiv \frac{1}{\gamma} \bar{\eta}_t. \quad (\text{C.10})$$

Using this we get the diffusion process of  $z_t \equiv \left(1 + \lambda_t^{1/\gamma}\right)^\gamma$ , as follows.

$$\frac{dz_t}{z_t} = \left[ \frac{1}{2} \gamma (\gamma - 1) \left( \frac{x_t}{1 + x_t} \right)^2 \alpha_1(t) + \gamma \left( \frac{x_t}{1 + x_t} \right) \alpha_2^2(t) \right] dt + \gamma \left( \frac{x_t}{1 + x_t} \right) \alpha_2(t) dB_I^{(2)} \quad (\text{C.11})$$

,where  $x_t \equiv \lambda_t^{1/\gamma}$ . For mathematical tractability we approximate this process by simplifying  $x_t/(1+x_t)$ . Belief difference,  $\eta_t$  determines the process of Radon-Nikodym derivative  $\lambda_t$ . In our quantitative study we use 0.02 to 0.2 of  $\eta_t$  for firms in value decile. Simulation shows that  $x_t/(1+x_t)$  are very similar to 0.5. By plugging  $\alpha_1(t)$  and  $\alpha_2(t)$  into the equation above and using the approximation that  $x_t/(1+x_t) \approx$

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<sup>2</sup>The assumption on the process  $H_t$  is very similar to the one in Santos and Veronesi (2010).

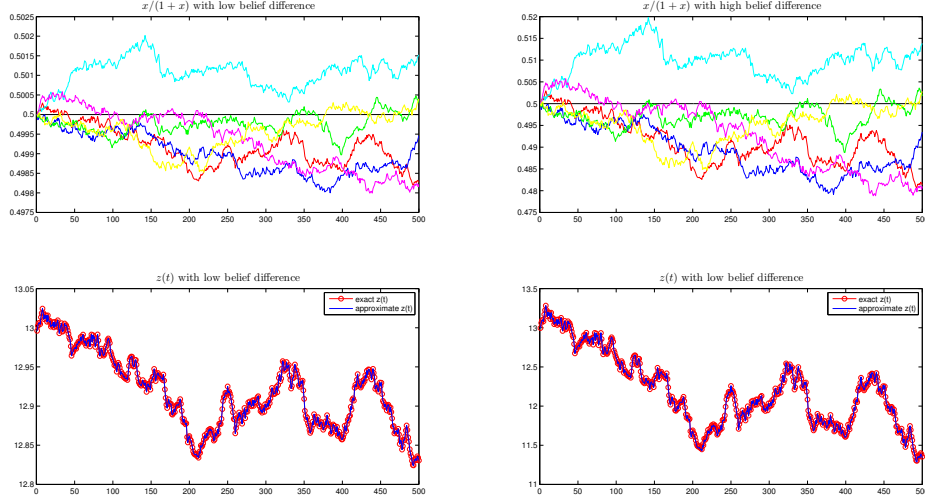
1/2(see Figure below), we have an approximate process of  $z_t$  as follows.

$$\frac{dz_t}{z_t} \approx \tilde{\alpha}_1(t)dt + \tilde{\alpha}_2(t)dB_I^{(2)}, \quad (\text{C.12})$$

where

$$\begin{aligned} \tilde{\alpha}_1(t) &\equiv -\frac{1}{8} \left(1 - \frac{1}{\gamma}\right) \bar{\eta}_t^2, \\ \tilde{\alpha}_2(t) &\equiv \frac{1}{2} \bar{\eta}_t. \end{aligned} \quad (\text{C.13})$$

From now on, we use this approximate  $z_t$  process for the rest of the proof. In order



to get the diffusion process of  $q_t$ , we use the diffusion process of  $z_t H_t$ ;

$$\frac{d(z_t H_t)}{z_t H_t} = \mu_{zH} dt + h_2 dB_A + \tilde{\alpha}_2 dB_I^{(2)}, \quad (\text{C.14})$$

where  $\mu_{zH} \equiv \tilde{\alpha}_1 + h_1 [\bar{H}/H_t - 1]$ . By using this process, we get the diffusion process

of  $q_t$  as follows.

$$\frac{dq_t}{q_t} = \mu_q(t)dt + (\sigma_{s,A}(t) + h_2) dB_A + (\tilde{\alpha}_2(t) + \sigma_{s,I}(t)) dB_I^{(2)}, \quad (\text{C.15})$$

where

$$\mu_q(t) \equiv \tilde{\alpha}_1(t) + h_1 \left( \frac{\bar{H}}{H_t} - 1 \right) + \phi_s \left( \frac{\bar{s}^{(2)}}{s_t} - 1 \right) + h_2 \sigma_{s,A}(t) + \tilde{\alpha}_2(t) \sigma_{s,I}(t)$$

. In drift term of  $dq_t$ ,  $q_t \mu_q(t)$ , we have terms of  $q_t$ ,  $s_t z_t$  and  $z_t H_t$  as

$$q_t \mu_q(t) \equiv (\tilde{\alpha}_1(t) + \tilde{\alpha}_2(t) \sigma_{s,I}(t) + h_2 \sigma_{s,A}(t) - h_1 - \phi_s) [q_t] + h_1 \bar{H} [s_t z_t] + \phi_s \bar{s}^{(2)} [z_t H_t]. \quad (\text{C.16})$$

Note that

$$\begin{aligned} d(s_t z_t) &= \{ (\tilde{\alpha}_1(t) - \phi_s + \tilde{\alpha}_2(t) \sigma_{s,I}(t)) [s_t z_t] + \phi_s \bar{s}^{(2)} [z_t] \} dt + [\dots] dB_A + [\dots] dB_I^{(2)}, \\ d(z_t H_t) &= \{ (\tilde{\alpha}_1(t) - h_1) [z_t H_t] + h_1 \bar{H} [z_t] \} dt + [\dots] dB_A + [\dots] dB_I^{(2)}. \end{aligned} \quad (\text{C.17})$$

We have  $z_t$ ,  $q_t$ ,  $s_t z_t$ , and  $z_t H_t$  variables in the diffusion processes of variables in the drift of  $dq_t$ . Thus we take a vector process  $y_t \equiv [z_t, q_t, s_t z_t, z_t H_t]'$  for computing equilibrium price-dividend ratio.  $y_t$  follows a diffusion process as follows.

$$dy_t = Y_1 y_t dt + \Sigma(y_t) d\mathbf{B}'^{(2)}, \quad (\text{C.18})$$

where  $\Sigma(y_t)$  is the appropriate matrix of diffusion coefficients,  $Y_1 \equiv [y_{ij}]_{4 \times 4}$  is the

matrix of drift coefficients;

$$\begin{pmatrix} \tilde{\alpha}_1(t) & 0 & 0 & 0 \\ 0 & \tilde{\alpha}_1(t) + \tilde{\alpha}_2(t)\sigma_{s,I}(t) + h_2\sigma_{sA}(t) - h_1 - \phi_s & h_1\bar{H} & \phi_s\bar{s}^{(2)} \\ \phi_s\bar{s}^{(2)} & 0 & \tilde{\alpha}_1(t) + \tilde{\alpha}_2(t)\sigma_{s,I}(t) - \phi_s & 0 \\ h_1\bar{H} & 0 & 0 & \tilde{\alpha}_1(t) - h_1 \end{pmatrix}. \quad (\text{C.19})$$

To avoid notational abuse, we denote  $Y_1$  as

$$\begin{pmatrix} y_{11} & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & y_{24} \\ y_{24} & 0 & y_{33} & 0 \\ y_{23} & 0 & 0 & y_{44} \end{pmatrix}, \quad (\text{C.20})$$

where  $y_{11} = y_{11}(\bar{\eta}_t)$ ,  $y_{22} = y_{22}(\bar{\eta}_t, s_t)$ ,  $y_{33} = y_{33}(\bar{\eta}_t, s_t)$ , and  $y_{44} = y_{44}(\bar{\eta}_t)$  since  $\sigma_{s,A}(t)$  and  $\sigma_{s,I}(t)$  are functions of  $s_t$ . Eigenvalues of the matrix,  $Y_1$ , are given by diagonal elements of  $Y_1$ . For mathematical tractability, we assume that all eigenvalues are negative following Menzley, Santos, and Veronesi (2004). Later our simulation study shows that all diagonal elements are indeed negative. Note that the conditional expectation  $E_t[q_{t+\tau}]$  is given by

$$\begin{aligned} E_t[q_{t+\tau}] &= e_2 E_t[y_{t+\tau}] \\ &= e_2 \Psi(\tau; t) y_t, \end{aligned} \quad (\text{C.21})$$

where  $e_2 \equiv (0, 1, 0, 0)$ ,

$$\Psi(\tau; t) = U \exp(\Lambda \cdot \tau) U^{-1}, \quad (\text{C.22})$$

$\Lambda$  is the diagonal matrix with its elements being eigenvalues of  $Y_1$  and  $U$  is the matrix of eigenvectors of  $Y_1$ . Note that eigenvalues of  $Y_1$  are diagonal elements of  $Y_1$ . Thus

$U$  is given by

$$U = \begin{pmatrix} u_{11} & 0 & 0 & 0 \\ u_{21} & 1 & u_{23} & u_{24} \\ u_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad (\text{C.23})$$

, where

$$\begin{aligned} u_{11} &= \frac{y_{11} - y_{44}}{y_{23}}, \\ u_{21} &= \frac{y_{24}(2y_{11} - y_{33} - y_{44})}{(y_{11} - y_{22})(y_{11} - y_{33})}, \\ u_{31} &= \frac{y_{31}(y_{11} - y_{44})}{y_{23}(y_{11} - y_{33})}, \\ u_{23} &= \frac{y_{23}}{y_{33} - y_{22}}, \\ u_{24} &= \frac{y_{24}}{y_{44} - y_{22}}. \end{aligned} \quad (\text{C.24})$$

The inverse matrix  $U^{-1} \equiv V = [v_{ij}]$  is given by

$$V = \begin{pmatrix} v_{11} & 0 & 0 & 0 \\ v_{21} & 1 & v_{23} & v_{24} \\ v_{31} & 0 & 1 & 0 \\ v_{41} & 0 & 0 & 1 \end{pmatrix}, \quad (\text{C.25})$$

where

$$\begin{aligned}
v_{11} &= 1/u_{11}, \\
v_{21} &= \frac{-u_{21} + u_{24} + u_{23}u_{31}}{u_{11}}, \\
v_{23} &= -u_{23}, \\
v_{24} &= -u_{24}, \\
v_{31} &= \frac{-u_{31}}{u_{11}}, \\
v_{34} &= -u_{34}, \\
v_{41} &= -1/u_{11}.
\end{aligned} \tag{C.26}$$

Using these quantities, we can compute  $\Psi(\tau)$  so that we get  $E_t[q_{t+\tau}]$  as following;

$$E_t[q_{t+\tau}] = e_2\Psi(\tau; t)y_t = \Psi_1(\tau; t)z_t + \Psi_2(\tau; t)q_t + \Psi_3(\tau; t)s_tz_t + \Psi_4(\tau; t)z_tH_t, \tag{C.27}$$

where

$$\begin{aligned}
\Psi_1(\tau; t) &= v_{11}u_{21}e^{y_{11}\tau} + v_{21}e^{y_{22}\tau} + v_{31}u_{23}e^{y_{33}\tau} + v_{41}u_{24}e^{y_{44}\tau}, \\
\Psi_2(\tau; t) &= e^{y_{22}\tau}, \\
\Psi_3(\tau; t) &= v_{23}e^{y_{22}\tau} + u_{23}e^{y_{33}\tau}, \\
\Psi_4(\tau; t) &= v_{24}e^{y_{22}\tau} + u_{24}e^{y_{44}\tau}.
\end{aligned} \tag{C.28}$$



Therefore

$$\begin{aligned}
E_t \left[ \int_0^\infty q_{t+\tau} d\tau \right] &= \int_0^\infty E_t [q_{t+\tau}] d\tau \\
&= \int_0^\infty e_2 \Psi(\tau; t) y_t d\tau \\
&= \sum_{k=1}^4 \left[ \int_0^\infty \Psi_k(\tau; t) d\tau \right] y_k(t),
\end{aligned} \tag{C.29}$$

where  $y_k(t)$  is the  $k$ -th row vector  $y_t$ .  $\int_0^\infty \Psi_k(\tau; t)$ 's are given by

$$\begin{aligned}
\int_0^\infty \Psi_1(\tau; t) d\tau &= \left[ -\frac{v_{11}u_{21}}{y_{11}} - \frac{v_{21}}{y_{22}} - \frac{v_{31}u_{23}}{y_{33}} - \frac{v_{41}u_{24}}{y_{44}} \right], \\
\int_0^\infty \Psi_2(\tau; t) d\tau &= -\frac{1}{y_{22}}, \\
\int_0^\infty \Psi_3(\tau; t) d\tau &= -\frac{v_{23}}{y_{22}} - \frac{u_{23}}{y_{33}}, \\
\int_0^\infty \Psi_4(\tau; t) d\tau &= -\frac{v_{24}}{y_{22}} - \frac{u_{24}}{y_{44}}.
\end{aligned} \tag{C.30}$$

Integrations above were conducted under the assumption that all the eigenvalues are negative as assumed before. Thus the approximated equilibrium stock price with the

share process  $s_t$  is given by

$$\begin{aligned}
P_s(t) &\approx \frac{D_s(t)}{q_t} \sum_{k=1}^4 \left[ \int_0^\infty \Psi_k(\tau; t) d\tau \right] y_k(t) \\
&= D_s(t) \sum_{k=1}^4 \left[ \int_0^\infty \Psi_k(\tau; t) d\tau \right] \left[ \frac{y_k(t)}{q_t} \right] \\
&= D_s(t) \left\{ \left[ \int_0^\infty \Psi_1(\tau; t) d\tau \right] \frac{1}{s_t H_t} + \left[ \int_0^\infty \Psi_2(\tau; t) d\tau \right] + \left[ \int_0^\infty \Psi_3(\tau; t) d\tau \right] \frac{1}{H_t} \right. \\
&\quad \left. + \left[ \int_0^\infty \Psi_4(\tau; t) d\tau \right] \frac{1}{s_t} \right\} \\
&= D_s(t) \left\{ \left[ \int_0^\infty \Psi_2(\tau; t) d\tau \right] + \left[ \int_0^\infty \frac{\Psi_1(\tau; t)}{\bar{s}^{(2)}} d\tau \right] \frac{\bar{s}^{(2)}}{s_t} H_t^{-1} + \left[ \int_0^\infty \Psi_3(\tau; t) d\tau \right] H_t^{-1} \right. \\
&\quad \left. + \left[ \int_0^\infty \frac{\Psi_4(\tau; t)}{\bar{s}^{(2)}} d\tau \right] \frac{\bar{s}^{(2)}}{s_t} \right\} \\
&= D_s(t) \left[ \beta_{0,t} + \beta_{1,t} \left( \frac{\bar{H}}{H_t} \right) + \beta_{2,t} \left( \frac{\bar{s}^{(2)}}{s_t} \right) \right. \\
&\quad \left. + \beta_{3,t} \left( \frac{\bar{s}^{(2)}}{s_t} \frac{1}{H_t} \right) \right],
\end{aligned} \tag{C.31}$$

where  $\beta_j$ 's are

$$\begin{aligned}
\beta_{0,t} &\equiv \int_0^\infty \Psi_2(\tau; t) d\tau, \quad \beta_{1,t} \equiv \int_0^\infty \frac{\Psi_3(\tau; t)}{\bar{H}} d\tau, \\
\beta_{2,t} &\equiv \int_0^\infty \frac{\Psi_4(\tau; t)}{\bar{s}^{(2)}} d\tau, \quad \beta_{3,t} \equiv \int_0^\infty \frac{\Psi_1(\tau; t)}{\bar{s}^{(2)} \bar{H}} d\tau.
\end{aligned} \tag{C.32}$$

The equilibrium price-dividend ratio of the shared stock is, hence, given by

$$\frac{P_s(t)}{D_s(t)} = \left[ \beta_{0,t} + \beta_{1,t} \left( \frac{\bar{H}}{H_t} \right) + \beta_{2,t} \left( \frac{\bar{s}^{(2)}}{s_t} \right) + \beta_{3,t} \left( \frac{\bar{s}^{(2)}}{s_t} \frac{\bar{H}}{H_t} \right) \right]. \tag{C.33}$$

□

**Proof of Proposition 2.** First we find diffusion coefficients of  $\frac{dP_{s_t}}{P_{s_t}}$  in the perspec-

tive of investor 2. Applying Ito lemma to  $P_s(t)$  that was derived in the previous Proposition, we get diffusion coefficients of  $\frac{dP_{s_t}}{P_{s_t}}$  as follows<sup>3</sup>.

$$\begin{aligned} dB_A : & \quad \left(\frac{D_s}{P_s}\right) \beta_{0,t} (\sigma_{D,A} + \sigma_{s,A}(t)) + \left(\frac{D_s}{P_s}\right) \beta_{1,t} (\sigma_{D,A} + \sigma_{s,A}(t) - h_2) \frac{\bar{H}}{H_t} \\ & + \left(\frac{D_s}{P_s}\right) \beta_{2,t} \sigma_{D,A} \left(\frac{\bar{s}^{(2)}}{s_t}\right) + \left(\frac{D_s}{P_s}\right) \beta_{3,t} (\sigma_{D,A} - h_2) \left(\frac{\bar{s}^{(2)}}{s_t} \frac{1}{H_t}\right), \quad (C.34) \\ dB_I^{(2)} : & \quad \left(\frac{D_s}{P_s}\right) \sigma_{s,I}(t) \left(\beta_{0,t} + \beta_{1,t} \frac{\bar{H}}{H_t}\right). \end{aligned}$$

The diffusion coefficients of shared asset's excess return (defined as  $R_s$  and  $dR_s \equiv \frac{dP_{s_t} + D_s(t)}{P_{s_t}} - r_t dt$ ) is the same as ones in  $\frac{dP_{s_t}}{P_{s_t}}$ . Note that in equilibrium the expected excess stock return  $E_t[dR_s]$  is given by the negative of the inner product of the diffusion coefficient vector of  $dR_s$  and the diffusion coefficient vector of the state price density  $\xi_t^{(2)}$  since equilibrium return is defined by the covariance between aforementioned two quantities. Applying Ito lemma to  $\xi_t^{(2)} = (1 + \lambda_t^{1/\gamma})^\gamma (1/X_t)^{1-\gamma} D_t^{-\gamma} = z_t H_t D_t^{-1}$  using the approximate diffusion process of  $z_t$ , gives

$$d\xi_t^{(2)}/\xi_t^{(2)} = \mu_{\xi^{(2)}} dt + (h_2 - \sigma_{D,A}) dB_A + \tilde{\alpha}_2(t) dB_I^{(2)}, \quad (C.35)$$

where  $\mu_{\xi^{(2)}} = \tilde{\alpha}_1(t) + h_1(\bar{H}/H_t - 1) + \sigma_{D,A}^2 - \mu_D - h_2\sigma_{D,A}$ . Since the expected excess return is determined by the negative of the sum of multiplications of diffusion

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<sup>3</sup>Here we simplify the computation by not applying Ito lemma to coefficients,  $\beta_{k,t}$ 's. Note that  $\sigma_{s,A}$  and  $\sigma_{s,I}$  are time-varying since they depend on  $s_t$ . These two parameters are estimated using underlying parameters such as  $v_{s,A}$  and  $v_{s,I}$ . The estimation of these two parameters shows that  $\sigma_{s,A}$  and  $\sigma_{s,I}$  are almost the same as  $v_{s,A}$  and  $v_{s,I}$  respectively without much variations in their time-series from 1983 to 2011. Thus we take  $\sigma_{s,A}$  and  $\sigma_{s,I}$  as constants when applying Ito lemma to equilibrium price-dividend ratio. This greatly reduces computational complexities. Menzley, Santos, and Veronesi (2004) used the same method when they compute the excess return process of an asset with share  $s_t$ . More specifically they used the identification equation for the share process so that  $\sigma_{s,A}(t)$  and  $\sigma_{s,I}(t)$  are approximated by  $v_{s,A}$  and  $v_{s,I}$  respectively. This approximation cannot be done when  $\sigma_{s,A}(t)$  and  $\sigma_{s,I}(t)$  are very different from  $v_{s,A}$  and  $v_{s,I}$ , which justifies our computation.

coefficients given in the equation (C.34) with  $\sigma_{D,A} - h_2$  and  $-\tilde{\alpha}_2(t)$  respectively,

$$E_t[dR_s] \approx \left[ \frac{D_s(t)}{P_s(t)} \right] \left[ \mu_{s,t}^{A,I} + \mu_{s,t}^I \right], \quad (\text{C.36})$$

where

$$\begin{aligned} \mu_{s,t}^{A,I} &\equiv \beta_{0,t} (\sigma_{D,A} + \sigma_{s,A}(t)) (\sigma_{D,A} - h_2) + \beta_{1,t} (\sigma_{D,A} + \sigma_{s,A}(t) - h_2) (\sigma_{D,A} - h_2) \frac{1}{H_t} \\ &\quad + \beta_{2,t} \sigma_{D,A} (\sigma_{D,A} - h_2) \frac{\bar{s}^{(2)}}{s_t} + \beta_{3,t} (\sigma_{D,A} - h_2)^2 \frac{\bar{s}^{(2)}}{s_t} \frac{\bar{H}}{H_t}, \\ \mu_{s,t}^I &\equiv -\sigma_{s,I}(t) \tilde{\alpha}_2(t) \left( \beta_{0,t} + \beta_{1,t} \frac{\bar{H}}{H_t} \right). \end{aligned} \quad (\text{C.37})$$

As was already shown above, the diffusion process of the return of an shared asset,  $R_s(t)$  is given by

$$dR_s = \mu_{R_s} dt + \sigma_{R_s,A} dB_A + \sigma_{R_s,I} dB_I^{(2)}, \quad (\text{C.38})$$

where  $\mu_{R_s}$  is the expected excess return given above and both  $\sigma_{R_s,A}$  and  $\sigma_{R_s,I}$  are diffusion coefficients of  $dP_s/P_s$  given above. From the diffusion process of stochastic discount factor above, we can retrieve the approximate equilibrium interest rate as the negative of the drift  $\mu_{\xi^{(2)}}$ . Thus we have

$$r_t \approx \mu_D - \sigma_{D,A}^2 - h_1 \left( \frac{\bar{H}}{H_t} - 1 \right) + h_2 \sigma_{D,A} - \tilde{\alpha}_1(t). \quad (\text{C.39})$$

Note that this expression permits similar interpretations of the original equilibrium interest rate in the exact form. It depends negatively on the aggregate volatility and positively on the drift of the aggregate cash flow process. Also as it has  $\alpha_1$ , we have the effects of the belief difference and the idiosyncratic risk. Note that the effect of the idiosyncratic risk comes from the optimal consumption choice of investors.  $\square$

***Proof of Proposition 3.*** The equilibrium price of the market portfolio can be obtained as a special case of the equilibrium price of the individual shared asset. Thus the equilibrium aggregate price dividend ratio is given by

$$\frac{P_t}{D_t} = \frac{1}{h_1} - \frac{\bar{H}}{H_t}. \quad (\text{C.40})$$

The diffusion coefficient of  $dP_t/P_t$  is given by

$$(\sigma_{D,A} - h_2) + \frac{h_2}{h_1} \left( \frac{D_t}{P_t} \right). \quad (\text{C.41})$$

Aggregate expected excess return is given by the negative of product of the diffusion coefficient above and the aggregate market price of risk.

$$E_t[dR_t] = (\sigma_{D,A} - h_2)^2 + \frac{h_2}{h_1} (\sigma_{D,A} - h_2) \frac{D_t}{P_t}. \quad (\text{C.42})$$

And the diffusion coefficient of the aggregate excess return  $R_t$  is given by

$$dR_t = \mu_R dt + \sigma_{R,A} dB_A, \quad (\text{C.43})$$

where  $\mu_{R_t}$  is the expected excess return given above and  $\sigma_{R_t,A}$  is the diffusion coefficient given above.

□

## APPENDIX D

### ADDITIONAL EQUILIBRIUM RESULTS

Expressions in the previous appendix together with the equation (2.18) yields the equilibrium consumption sharing rules for investors as follows.

**Proposition 5.** *The equilibrium consumption sharing rules are given by*

$$\begin{aligned} c_2(t) &= \omega(\lambda_t) D_t, \\ c_1(t) &= (1 - \omega(\lambda_t)) D_t, \end{aligned} \tag{D.1}$$

where  $\omega(\lambda_t) \equiv \left[ 1 + \left( \frac{1}{\lambda_t} \right)^{-\frac{1}{\gamma}} \right]^{-1}$ .

The diffusion processes of consumptions for investors  $k=1,2$  are given by

$$\frac{dc_k(t)}{c_k(t)} = \mu_{c_k}^{(k)} dt + \sigma_{c_k,A}(t) dB_A + \sigma_{c_k,I}(t) dB_I^{(k)}, \tag{D.2}$$

where

$$\begin{aligned} \mu_{c_2}^{(2)} &= \mu_D - \frac{1}{2} \frac{1}{\gamma} \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}} \right) \bar{\eta}_t^2 + \frac{1}{\gamma^2} \left( \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}} \right)^2 \bar{\eta}_t^2, \\ \mu_{c_1}^{(1)} &= \left( 1 + \frac{c_2(t)}{c_1(t)} \right) \mu_D - \frac{c_2(t)}{c_1(t)} \mu_{c_2}^{(2)} - \frac{c_2(t)}{c_1(t)} \bar{\eta}_t \sigma_{c_2,I}(t), \\ \sigma_{c_1,A} &= \sigma_{c_2,A} = \sigma_D, \\ \sigma_{c_2,I}(t) &= - \left( \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}} \right) \left( \frac{1}{\gamma} \right) \bar{\eta}_t, \\ \sigma_{c_1,I}(t) &= - \left( \frac{c_2}{c_1} \right) \sigma_{c_2,I} = - \left( \frac{1}{\lambda_t^{1/\gamma}} \right) \sigma_{c_2,I}(t). \end{aligned} \tag{D.3}$$

**Proof of Proposition 5.** With the consumption goods market clearing equation,

we have the following;

$$\left(\frac{1}{X_t}\right)^{\frac{1-\gamma}{\gamma}} \left[ \left(\frac{1}{\lambda_2 \lambda_t}\right)^{-\frac{1}{\gamma}} + \left(\frac{1}{\lambda_2}\right)^{-\frac{1}{\gamma}} \right] \xi_t^{(2)-\frac{1}{\gamma}} = D_t. \quad (\text{D.4})$$

By arranging this equation we get  $\xi_t^{(2)}$ . Plugging this back into the equation (2.18) yields the required expression for the consumption sharing rules. By applying Ito lemma to optimal consumption sharing rules, we get

$$\begin{aligned} \frac{dc_1(t)}{c_1(t)} &= \mu_{c_1}^{(1)} dt + \sigma_{c_1,A} dB_A + \sigma_{c_1,I} dB_I^{(1)} \\ &= \mu_{c_1} dt + \sigma_{c_1,A} dB_A + \sigma_{c_1,I} dB_I, \\ \frac{dc_2(t)}{c_2(t)} &= \mu_{c_2}^{(2)} dt + \sigma_{c_2,A} dB_A + \sigma_{c_2,I} dB_I^{(2)} \\ &= \mu_{c_2} dt + \sigma_{c_2,A} dB_A + \sigma_{c_2,I} dB_I, \end{aligned} \quad (\text{D.5})$$

where

$$\begin{aligned}
\mu_{c_2}^{(2)} &= \mu_D - \frac{1}{2} \frac{1}{\gamma} \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}} \right) \bar{\eta}_t^2 + \frac{1}{\gamma^2} \left( \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}} \right)^2 \bar{\eta}_t^2, \\
\mu_{c_2} &= \mu_{c_2}^{(2)} + \frac{1}{\gamma} \left( \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}} \right) \bar{\eta}_t \eta_t^{(2)}, \\
\mu_{c_1}^{(1)} &= \left( 1 + \frac{c_2(t)}{c_1(t)} \right) \mu_D - \frac{c_2(t)}{c_1(t)} \mu_{c_2}^{(2)} - \frac{c_2(t)}{c_1(t)} \bar{\eta}_t \sigma_{c_2, I}(t), \\
&= \left( 1 + \frac{1}{\lambda_t^{1/\gamma}} \right) \mu_D - \frac{1}{\lambda_t^{1/\gamma}} [\mu_{c_2}^{(2)} + \bar{\eta}_t \sigma_{c_2, I}(t)], \\
\mu_{c_1} &= \mu_{c_1}^{(1)} - \frac{1}{\gamma} \frac{1}{\lambda_t^{1/\gamma}} \left( \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}} \right) \bar{\eta}_t \eta_t^{(1)},
\end{aligned} \tag{D.6}$$

$$\begin{aligned}
\sigma_{c_1, A} &= \sigma_{c_2, A} = \sigma_{D, A}, \\
\sigma_{c_2, I}(t) &= - \left( \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}} \right) \left( \frac{1}{\gamma} \right) \bar{\eta}_t, \\
\sigma_{c_1, I}(t) &= - \left( \frac{c_2(t)}{c_1(t)} \right) \sigma_{c_2, I}(t) = - \left( \frac{1}{\lambda_t^{1/\gamma}} \right) \sigma_{c_2, I}(t).
\end{aligned}$$

□

As conventional wisdom says, consumptions sharing rules are proportional to the aggregate dividend. However it also depends on the  $\lambda_t$  process that reflects the belief difference associated with the idiosyncratic risk. Dependence on the idiosyncratic risk (via  $\lambda_t$ ) of consumption sharing rules is different from the conventional result. In existing asset pricing studies, idiosyncratic risks do not affect equilibrium quantities since they are diversified away. Existing literature on idiosyncratic risk has focused on non-diversifiable consumers' idiosyncratic income risk. Kahn (1990), Franke, Stapleton, and Subrahmanyam (1993), Telmer (1993), Aiyagari (1994), Lucas (1994), Heaton and Lucas (1996), and Storesletten, Telmer, and Yaron (2007) show that idiosyncratic income risk affects equilibrium quantities. However these studies do



not really investigate financial idiosyncratic risk. Our specific mechanism of heterogeneous beliefs leads to above result since idiosyncratic cash flow risk is priced in equilibrium. The weight in optimal consumption sharing rule,  $\omega(\lambda_t)$ , shows that not only aggregate systematic risk plays a role, but also idiosyncratic cash flow risk plays an important role through investors' belief differences. Thus our result might be the first that equilibrium consumption choices can be affected by idiosyncratic risk.

The equilibrium interest rate and the market price of risks across investors are given below.

**Proposition 6.** *The equilibrium interest rate,  $r_t$ , is given by*

$$\begin{aligned} r(t) &= A(t) \left[ \sum_{k=1}^2 c_k(t) \mu_{c_k}^{(k)} - \frac{1}{2} \sum_{k=1}^2 P_k(t) c_k(t)^2 (\sigma_{c_k,A}^2 + \sigma_{c_k,I}^2) \right] \\ &= \gamma \mu_D - \frac{1}{2} \gamma (1 + \gamma) \sigma_{D,A}^2 - \frac{1}{2} \left( \frac{1 + \gamma}{\gamma} \right) \left( \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}} \right)^2 \bar{\eta}_t^2, \end{aligned} \quad (\text{D.7})$$

where  $P_k$  is the prudence parameter of investor  $k$ , and  $A$  is the aggregate absolute risk aversion parameter.

The parametric expressions of the aggregate market price of risk and market prices of the idiosyncratic risks across investors are given by

$$\begin{aligned} \theta_A(t) &= \gamma \sigma_{D,A}, \\ \theta_I^{(2)}(t) &= -\bar{\eta}_t \left( \frac{A(t)}{A_1(t)} \right) = -\bar{\eta}_t \frac{c_1(t)}{D_t} = -\bar{\eta}_t (1 - \omega(\lambda_t)), \\ \theta_I^{(1)}(t) &= \bar{\eta}_t \left( \frac{A(t)}{A_2(t)} \right) = \bar{\eta}_t \frac{c_2(t)}{D_t} = \bar{\eta}_t \omega(\lambda_t), \end{aligned} \quad (\text{D.8})$$

where  $A_k$  is the investor  $k$ 's absolute risk aversion parameter.

**Proof of Proposition 6.** First we define some quantities below for the proof. These are parameters of individual investor's absolute risk aversion and prudence,

$A_i$  and  $P_i$  and of representative investor's absolute risk aversion and prudence,  $A$  and  $P$ .

$$\begin{aligned}
A_k(t) &= -\frac{u''(c_k(t))}{u'(c_k(t))} = \frac{\gamma}{c_k(t)}, \\
P_k(t) &\equiv -\frac{u'''(c_k(t))}{u''(c_k(t))} = \frac{1+\gamma}{c_k(t)}, \\
A(t) &\equiv -\frac{U''(C(t))}{U'(C(t))} = \frac{1}{1/A_1(t) + 1/A_2(t)} = \frac{\gamma}{c_1(t) + c_2(t)} = \frac{\gamma}{D_t}, \\
P(t) &\equiv -\frac{U'''(C(t))}{U''(C(t))} = \sum_{k=1}^2 \left( \frac{A(t)}{A_k(t)} \right)^2 P_k(t) = \frac{1+\gamma}{D_t}.
\end{aligned} \tag{D.9}$$

By applying Ito lemma to the first order condition of the individual optimization problem,  $u'(c_k) = \frac{\xi^{(k)}}{\lambda_k}$ , we have the following conditions

$$\begin{aligned}
-\frac{\xi_t^{(k)}}{\lambda_k} r_t &= c_k(t) u''(c_k(t)) \mu_{c_k}^{(k)} + \frac{1}{2} c_k(t)^2 u'''(c_k(t)) (\sigma_{c_k,A}^2 + \sigma_{c_k,I}^2), \\
-\frac{\xi_t^{(k)}}{\lambda_k} \theta_A &= c_k(t) u''(c_k(t)) \sigma_{c_k,A}, \\
-\frac{\xi_t^{(k)}}{\lambda_k} \theta_I^{(k)} &= c_k(t) u''(c_k(t)) \sigma_{c_k,I},
\end{aligned} \tag{D.10}$$

by matching coefficients of  $dt$ ,  $dB_A$  and  $dB_I^{(k)}$  respectively. Also by applying Ito lemma to the consumption goods market clearing condition with true idiosyncratic Brownian motion,  $B_I(t)$ , we get the following conditions

$$\begin{aligned}
c_1 \mu_{c_1}^{(1)} + \mu_{c_2}^{(2)} &= D \mu_D, \\
D \sigma_{D,A} &= c_1 \sigma_{c_1,A} + c_2 \sigma_{c_2,A}, \\
c_1 \sigma_{c_1,I} + c_2 \sigma_{c_2,I} &= 0,
\end{aligned} \tag{D.11}$$

by matching coefficients of  $dt$ ,  $dB_A$ , and  $dB_I$  (instead of  $B_I^{(k)}$ ) respectively. Thanks to conditions above and the first order condition of the individual optimization problem,

the first equation in (D.10) becomes

$$\frac{r_t}{A_k} = c_k \mu_{c_k}^{(k)} - \frac{1}{2} P_k c_k^2 (\sigma_{c_k, A}^2 + \sigma_{c_k, I}^2). \quad (\text{D.12})$$

By summing this equation across investors, we get the required equilibrium interest rate in the first equality. For the second equality, we can use definitions of  $A_t$ ,  $A_k$ ,  $P_k$ ,  $P_t$  and the diffusion coefficient of equilibrium consumption sharing rule to get the required results.

Using second equations in both (D.10) and (D.11), we can deduce the expression of the market price of the aggregate risk. Also by using third equations in both (D.10) and (D.11), we obtain following condition;

$$\begin{aligned} c_1 \sigma_{c_1, I} + c_2 \sigma_{c_2, I} &= \frac{\theta_I^{(1)}}{A_1} + \frac{\theta_I^{(2)}}{A_2} \\ &= \frac{\theta_I - \eta^{(1)}}{A_1} + \frac{\theta_I^{(2)}}{A_2} \\ &= \frac{\theta_I^{(2)} + \eta^{(2)} - \eta^{(1)}}{A_1} + \frac{\theta_I^{(2)}}{A_2} \\ &= \theta_I^{(2)} \left[ \frac{1}{A_1} + \frac{1}{A_2} \right] + \frac{\bar{\eta}}{A_1} \\ &= 0. \end{aligned} \quad (\text{D.13})$$

Thus we have  $\theta_I^{(2)}$  from the fourth equation. Also by using the relation  $\frac{\theta_I^{(1)}}{A_1} - \frac{\theta_I^{(2)}}{A_2} = 0$ , we can also get the expression of  $\theta_I^{(1)}$ . Parametric expressions follow from definitions and the optimal consumption sharing rules.  $\square$

The first term in the equilibrium interest rate reveals its dependence on the growth rate of the aggregate dividend. The second term is negative and stems from precautionary savings motive. The sum of first two terms are very similar to the equilibrium

interest rate in traditional Lucas type asset pricing models. However, the third term reveals an interesting feature. It says that the idiosyncratic consumption risk which is equivalent to the idiosyncratic risk that stems from the heterogeneous beliefs, actually plays a role as a risk factor. One crucial assumption in our model is that the investors have belief differences about the growth rates of any individual cash flows through the idiosyncratic Brownian motion, not through the aggregate Brownian motion. This leads to the survival of the idiosyncratic risk in the stochastic discount factors, which implies that the pricing of riskless assets can be affected by idiosyncratic risk. Since the value of  $\lambda_t$  is positive as it is a stochastic exponential, the sign of the extra term is unambiguously negative. This indicates that idiosyncratic risk will lower the equilibrium short rate compared to the one in the conventional asset pricing model. Similar arguments can be found in other literatures such as Basak (2000), Basak (2005) and Gallmeyer and Hollifield (2008). Since investors are facing extra volatility in equilibrium consumptions, they will increase their precautionary savings so that the riskless rate decreases.